2024

External Direct Sum Invariant Subspace and Decomposition of Coupled Differential-Difference Equations

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Abstract—**This article discusses the invariant subspaces that are restricted to be external direct sums. Some existence conditions are presented that facilitate finding such invariant subspaces. This problem is related to the decomposition of coupled differential-difference equations, leading to the possibility of lowering the dimensions of coupled differential-difference equations. As has been well documented, lowering the dimension of coupled differential-difference equations can drastically reduce the computational time needed in stability analysis when a complete quadratic Lyapunov-Krasovskii functional is used. Most known ad hoc methods of reducing the order are special cases of this formulation.**

I. Introduction

In this article, we consider invariant subspace of a special structure. Specifically, given matrices $A \in \mathbb{R}^{m\times m}$, $B \in \mathbb{R}^{m\times n}$, $C \in \mathbb{R}^{n\times m}$, $D \in \mathbb{R}^{n\times n}$, form the matrix

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1}$$

We would like to investigate the existence of a nontrivial invariant subspace $\mathcal{W}$ of S (or $\mathcal{W}$ is S-invariant)

$$S\mathcal{W} \subset \mathcal{W}, \tag{2}$$

with the restriction that $\mathcal{W}$ is an external direct sum of a subspace $\mathcal{U} \subset \mathbb{R}^m$ and a subspace $\mathcal{V} \subset \mathbb{R}^n$,

$$\mathcal{W} = \mathcal{U} \oplus \mathcal{V}, \tag{3}$$

where the external direct sum $\oplus$ is defined as

$$\mathcal{U} \oplus \mathcal{V} := \left\{ \begin{pmatrix} u \\ v \end{pmatrix} : u \in \mathcal{U}, v \in \mathcal{V} \right\}. \tag{4}$$

Let the dimensions of $\mathcal{U}$ and $\mathcal{V}$ be

$$\dim(\mathcal{U}) = p, \tag{5}$$

$$\dim(\mathcal{V}) = q. \tag{6}$$

Then $0 \le p \le m$, $0 \le q \le n$. For the subspace $\mathcal{W}$ to be nontrivial, $0 < p+q < m+n$.

The existence of invariant subspace of such a structure arises from the possibility of decomposing coupled differential-difference equations

$$\dot{x}(t) = Ax(t) + By(t-r), \tag{7}$$

$$y(t) = Cx(t) + Dy(t-r) \tag{8}$$

into two sets of such equations of lower dimensions. In the above equations, $x \in \mathcal{X} = \mathbb{R}^m$ is the state variable for the differential equation, and $y \in \mathcal{Y} = \mathbb{R}^n$ is the state variable for the difference equation. Then the complete state space is $\mathcal{X} \times \mathcal{Y}$.

Coupled differential-difference equations arise in many practical problems in engineering and other scientific disciplines as reviewed in [12]. Early literature tends to formulate such a system in the general framework of differential-difference equations of neutral type[7]. The earliest example of direct stability analysis on such a system was given by Răsvan [11]. Various analysis of such a system has been conducted, see, for example, [1]. It was proposed in [4] that such a formulation have a substantial advantage even for differential-difference equations of retarded type when a complete type of Lyapunov-Krasovskii functional is used for stability analysis. It was also pointed out in [4] that systems with multiple commensurate delays may be rewritten as a coupled differential-difference equation with single delay. Indeed, several order of magnitude of saving of computational time have been reported in both the discretized Lyapunov-Krasovskii functional method [4] and sum-of-square method [15] due to reduced dimension of the delayed variable y as compared with the more traditional formulation (such as those given in [5] and [9]).

The method presented in this article will allow us to reduce the dimension of coupled differential-difference equations in a more systematic way. Many ad hoc methods used to reduce dimension of y such as the one presented in [4] turn out to be special cases of the method presented here. An important feature not explored before is the possibility of reducing not only the dimension of y, but also simultaneously that of x in the equations (7-8).

A preliminary version of this article was presented in [3]. This journal version added new examples to illustrate the procedure and expanded the discussions and references.

II. Conditions for component subspaces

In this section, we will discuss the requirements the component subspaces $\mathcal{U}$ and $\mathcal{V}$ need to satisfy in order for $\mathcal{W}$ to be an invariant subspace of S. We will first make the following simple observation.

Lemma 1. *The subspace $\mathcal{W}$ is S-invariant if and only if the component subspaces $\mathcal{U}$ and $\mathcal{V}$ satisfy*

$$A\mathcal{U} \subset \mathcal{U}, \tag{9}$$

$$B\mathcal{V} \subset \mathcal{U}, \tag{10}$$

$$C\mathcal{U} \subset \mathcal{V}, \tag{11}$$

$$D\mathcal{V} \subset \mathcal{V}. \tag{12}$$

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Proof. By definition, $\mathcal{W}$ is S-invariant if and only if all $u \in \mathcal{U}$ and $v \in \mathcal{V}$ satisfy

$$Au + Bv \in \mathcal{U}, \tag{13}$$

$$Cu + Dv \in \mathcal{V}. \tag{14}$$

It is easy to see that (9-12) are sufficient conditions for the above requirements. On the other hand, note that $0 \in \mathcal{U}$ and $0 \in \mathcal{V}$. Choose $v = 0$ in (13) and (14), we conclude that (9) and (11) are necessary conditions. Similarly, we can conclude that (10) and (12) are necessary by choosing $u = 0$ instead. ∎

Before presenting the main result, we would like to introduce the following definitions.

Definition 1. *For a matrix $E \in \mathbb{R}^{l\times l}$ and a subspace $\mathcal{F} \subset \mathbb{R}^l$, the reachable subspace (from the origin) of the pair $(E, \mathcal{F})$, which is denoted as $\langle E|\mathcal{F}\rangle$, is*

$$\langle E|\mathcal{F}\rangle = \bigcap\{\mathcal{H} : \mathcal{F} \subset \mathcal{H}, E\mathcal{H} \subset \mathcal{H}\}. \tag{15}$$

In the equation (15) above and the equation (16) below, $\mathcal{H}$ is restricted to be a subspace of $\mathbb{R}^l$. If $\langle E|\mathcal{F}\rangle = \mathbb{R}^l$, then the pair is said to be controllable. Otherwise, we say $(E, \mathcal{F})$ is uncontrollable.

For a pair of matrices $E \in \mathbb{R}^{l\times l}$ and $G \in \mathbb{R}^{h\times l}$, the unobservable subspace of the pair (G, E) is

$$\mathcal{N}(G, E) = \sum\{\mathcal{H} : \mathcal{H} \subset \mathrm{Ker}(G), E\mathcal{H} \subset \mathcal{H}\}. \tag{16}$$

In the above, $\sum$ means the sum of subspaces. The pair (G, E) is said to be observable if $\mathcal{N}(G, E) = \{0\}$. Otherwise, we say the pair is unobservable.

For the reachable subspace, it is well-known that [14]

$$\langle E|\mathcal{F}\rangle = \mathcal{F} + E\mathcal{F} + E^2\mathcal{F} + \cdots + E^{l-1}\mathcal{F}. \tag{17}$$

Alternatively, let F be a matrix such that

$$\mathrm{Im}(F) = \mathcal{F}.$$

Then [10]

$$\langle E|\mathcal{F}\rangle = \mathrm{span}\{F, EF, E^2F, \ldots, E^{l-1}F\}. \tag{18}$$

In the above, span refers to the subspace spanned by the column vectors of the matrices F, EF, etc.

For unobservable subspace, it is well known that [14]

$$\mathcal{N}(G, E) = \bigcap_{i=1}^{l} \mathrm{Ker}(GE^{i-1}). \tag{19}$$

Theorem 2. *Let S be defined in the equation (1). For a given subspace $\mathcal{V} \subset \mathbb{R}^n$ that is D-invariant, there exists a subspace $\mathcal{U} \subset \mathbb{R}^m$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ is S-invariant if and only if*

$$C\langle A|B\mathcal{V}\rangle \subset \mathcal{V}. \tag{20}$$

For such a $\mathcal{V}$, $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ is S-invariant if and only if $\mathcal{U}$ is A-invariant and satisfies

$$\langle A|B\mathcal{V}\rangle \subset \mathcal{U} \subset C^{-1}\mathcal{V}, \tag{21}$$

where $C^{-1}\mathcal{V} = \{u \in \mathbb{R}^m | Cu \in \mathcal{V}\}$.

Proof. It is sufficient to show that (9) and (21) constitute necessary and sufficient conditions for $\mathcal{W}$ to be S-invariant, or equivalently, for the satisfaction of the conditions (9-12). Note that (12) is satisfied by assumption.

To show necessity, note that conditions (9) and (10) implies $\mathcal{U} \supset \langle A|B\mathcal{V}\rangle$ in view of (15) in definition 1, which together with (11) implies (21).

To show sufficiency, note that (9) and (12) are satisfied by assumption. (21) obviously implies both (10) and (11).

Finally, for an A-invariant $\mathcal{U}$ that satisfies (21) to exist, (20) is obviously necessary. It is also sufficient because $\mathcal{U} = \langle A|B\mathcal{V}\rangle$ is A-invariant and satisfies (21). ∎

We can obtain the following corollary from the above theorem by symmetry of the role $\mathcal{U}$ and $\mathcal{V}$ play in this invariant subspace problem.

Corollary 3. *For a given subspace $\mathcal{U} \subset \mathbb{R}^m$ that satisfies (9), there exists a subspace $\mathcal{V} \subset \mathbb{R}^n$ such that $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ is S-invariant if and only if*

$$B\langle D|C\mathcal{U}\rangle \subset \mathcal{U}. \tag{22}$$

For such a $\mathcal{U}$, $\mathcal{W} = \mathcal{U} \oplus \mathcal{V}$ is S-invariant if and only if $\mathcal{V}$ is D-invariant and satisfies

$$\langle D|C\mathcal{U}\rangle \subset \mathcal{V} \subset B^{-1}\mathcal{U}. \tag{23}$$

III. Decomposition of coupled differential-difference equations

For the coupled differential-difference equations (7-8), carry out a variable transformation

$$x = Ux_1 + \hat{U}x_2, \tag{24}$$

$$y = Vy_1 + \hat{V}y_2, \tag{25}$$

where $x_1 \in \mathbb{R}^p$, $x_2 \in \mathbb{R}^{m-p}$, $y_1 \in \mathbb{R}^q$, $y_2 \in \mathbb{R}^{n-q}$, and the matrices $\begin{pmatrix} U & \hat{U} \end{pmatrix}$ and $\begin{pmatrix} V & \hat{V} \end{pmatrix}$ are nonsingular. We seek such a transformation so that the variables x_2 and y_2 are independent of x_1 and y_1. The transformed equations are

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}\begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \tag{26}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \tag{27}$$

where the coefficient matrices satisfy

$$A\begin{pmatrix} U & \hat{U} \end{pmatrix} = \begin{pmatrix} U & \hat{U} \end{pmatrix}\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}, \tag{28}$$

$$B\begin{pmatrix} V & \hat{V} \end{pmatrix} = \begin{pmatrix} U & \hat{U} \end{pmatrix}\begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix}, \tag{29}$$

$$C\begin{pmatrix} U & \hat{U} \end{pmatrix} = \begin{pmatrix} V & \hat{V} \end{pmatrix}\begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}, \tag{30}$$

$$D\begin{pmatrix} V & \hat{V} \end{pmatrix} = \begin{pmatrix} V & \hat{V} \end{pmatrix}\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}. \tag{31}$$

This article has been accepted for publication in IEEE Transactions on Automatic Control. This is the author's version which has not been fully edited and content may change prior to final publication. Citation information: DOI 10.1109/TAC.2023.3274208

For x_2 and y_2 to be independent of x_1 and y_1, we need

$$A_{21} = 0, B_{21} = 0, C_{21} = 0, D_{21} = 0. \tag{32}$$

Taking the first column block of the equations (28-31) and considering (32), we obtain

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} = \begin{pmatrix} U & 0 \\ 0 & V \end{pmatrix} \begin{pmatrix} A_{11} & B_{11} \\ C_{11} & D_{11} \end{pmatrix}. \tag{33}$$

Thus we reached the following conclusion.

Proposition 4. *The variable transformation (24-25) results in a system such that x_2 and y_2 are independent of x_1 and y_1 if and only if the matrices U and V satisfy (33) for some matrices A_{11}, B_{11}, C_{11} and D_{11} of appropriate dimensions.*

Let

$$\mathcal{U} = \mathrm{Im}(U), \tag{34}$$

$$\mathcal{V} = \mathrm{Im}(V). \tag{35}$$

Then (33) indicates that $\mathcal{U} \oplus \mathcal{V}$ is an invariant subspace of the matrix S defined in (1). Indeed, (33) is a matrix representation of the invariance relation (2).

A decomposed system consists of two subsystems with lower dimensions, and is more convenient to analyze than the original form both analytically and numerically. In the following, we will illustrate how the decomposition can drastically reduce the amount of computation in the stability analysis using the discretized Lyapunov-Krasovskii functional method developed in [4]. Consider a system (7-8) with

$$A = \begin{pmatrix} \frac{97}{3} & -78 & \frac{298}{3} & -50 \\ \frac{104}{3} & -75 & \frac{251}{3} & -42 \\ \frac{-4}{3} & -3 & \frac{11}{3} & -1 \\ -23 & 34 & -26 & 14 \end{pmatrix}, B = \begin{pmatrix} -16 & 4 & 13 \\ -19 & 2 & 18 \\ -16 & 1 & 18 \\ \frac{-43}{3} & \frac{7}{3} & \frac{50}{3} \end{pmatrix},$$

$$C = \begin{pmatrix} \frac{28}{3} & -17 & \frac{76}{3} & -15 \\ \frac{-17}{3} & 11 & \frac{-5}{3} & -2 \\ 6 & -11 & 18 & -11 \end{pmatrix}, D = \begin{pmatrix} \frac{8}{5} & \frac{13}{10} & \frac{-23}{5} \\ 4 & \frac{13}{10} & \frac{-37}{5} \\ 2 & \frac{7}{10} & \frac{-19}{5} \end{pmatrix}.$$

Carrying out the variable transformation (24-25) with

$$\begin{pmatrix} U & \hat{U} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 & 4 \\ 1 & 1 & 2 & 4 \\ 2 & 1 & 2 & 2 \\ 3 & 2 & 1 & 1 \end{pmatrix}, \begin{pmatrix} V & \hat{V} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 2 \\ 1 & 1 & 2 \end{pmatrix} \tag{36}$$

results in the following two decomposed subsystems:

$$\begin{aligned} \dot{x}_1(t) = & \begin{pmatrix} -1 & 2 \\ 2 & -8 \end{pmatrix} x_1(t) + \begin{pmatrix} 1 \\ 2 \end{pmatrix} y_1(t-r) \\ & + \begin{pmatrix} -3 & 5 \\ 9 & 1 \end{pmatrix} x_2(t) + \begin{pmatrix} -5 & -3 \\ 7 & 4 \end{pmatrix} y_2(t-r), \end{aligned} \tag{37}$$

$$\begin{aligned} y_1(t) = & \begin{pmatrix} -2 & -3 \end{pmatrix} x_1(t) + \begin{pmatrix} \frac{-2}{5} \end{pmatrix} y_1(t-r) \\ & + \begin{pmatrix} 4 & 7 \end{pmatrix} x_2(t) + \begin{pmatrix} 2 & 2 \end{pmatrix} y_2(t-r), \end{aligned} \tag{38}$$

and

$$\dot{x}_2(t) = \begin{pmatrix} -6 & -1 \\ 4 & -10 \end{pmatrix} x_2(t) + \begin{pmatrix} 0 & 1 \\ -4 & -5 \end{pmatrix} y_2(t-r), \tag{39}$$

$$y_2(t) = \begin{pmatrix} -1 & 2 \\ 3 & -2 \end{pmatrix} x_2(t) + \begin{pmatrix} \frac{1}{10} & -1 \\ \frac{1}{10} & \frac{-3}{5} \end{pmatrix} y_2(t-r). \tag{40}$$

The transformation does not affect the stability of the system. Therefore, the original system is stable if and only if the subsystem 2 consisting of (39-40) is stable, and the subsystem 1 consisting of (37-38) (with the terms involving x_2 and y_2 removed) is stable. It can be shown analytically, for example, using the method in [13], that the subsystem 1 is stable for $r \in [0, r_{1\max})$, where $r_{1\max} \approx 0.290583$, and it becomes unstable when the delay r increases beyond $r_{1\max}$. We will call $r_{1\max}$ the stability margin of the subsystem. Similarly, we can calculate that the stability margin of the subsystem 2 is $r_{2\max} \approx 0.163488$. Because $r_{2\max} < r_{1\max}$, the stability margin of the original system is $r_{2\max}$.

Then, we used the discretized Lyapunov-Krasovskii functional method developed in [4] with $N = 5$ to obtain estimated stability margins of subsystem 1, subsystem 2, and the original system. A PC with Windows environment running MATLAB with LMI toolbox [2] was used to do calculation. For each (sub-)system, we started with the initial interval of $[0, 5]$, and used a bisection method to determine an estimated stability margin until the interval size was less than 10^{-6}. The estimated stability margin and the computational time in CPU seconds are listed in the following table.

TABLE I
ESTIMATED STABILITY MARGINS & COMPUTATIONAL TIME

System	Estimated Stability Margin	CPU (second)
Subsystem 1	0.290581	7.33
Subsystem 2	0.163465	24.58
Original System	0.163465	106.67

From the above table, it can be seen that the stability margin estimated by the discrized Lyapunov-Krasovskii functional method is very close to the analytical results in all cases. The decomposition provided a substantial advantage in terms of computational time. If the decomposition is available, then the total computational time for the numerical stability analysis is $7.33 + 24.58 = 31.91$ seconds. However, if we conduct the numerical stability analysis directly on the original system, then the computational time would be 106.67 seconds, more than 3 times that for the decomposed case. The advantage of decomposition would be much more pronounced if the dimension of the system increases.

IV. Some special cases

We will first discuss a few special cases of the invariant subspace restricted to be external direct sum, and their interpretation in the context of coupled differential-difference equations.

A. *Uncontrollable difference equation*

Suppose the pair $(D, \mathrm{Im}(C))$ is uncontrollable. Let

$$\mathcal{V} = \langle D | \, \mathrm{Im}(C) \rangle = \mathrm{span}\{C, DC, D^2C, \ldots, D^{n-1}C\}. \tag{41}$$

Consider the conditions in Theorem 2. $\mathcal{V}$ is clearly D invariant. Furthermore, (20) is satisfied because

$$\mathrm{Im}(C) \subset \mathcal{V} \tag{42}$$

in view of (41). Equation (42) also implies $C^{-1}\mathcal{V} = \mathbb{R}^m$. Therefore, the condition (21) becomes

$$\langle A|B\mathcal{V}\rangle \subset \mathcal{U}. \tag{43}$$

In other words, $\mathcal{U}$ can be any A-invariant subspace that contains $\langle A|B\mathcal{V}\rangle$ (which is obviously A-invariant). The two extremes choices for $\mathcal{U}$ are $\langle A|B\mathcal{V}\rangle$ and $\mathbb{R}^m$. If we choose $\mathcal{U} = \mathbb{R}^m$, then $x_1 = x$, thus the variable x is not decomposed, resulting in a system where y_2 is independent of x and y_1.

A simple example is the differential-difference equation

$$\dot{x}(t) = Ax(t) + A_1 x(t-r). \tag{44}$$

As is commented in [4], even for this system of retarded type, there is an advantage of writing it in the form of couple differential-difference equations when A_1 is singular. This is due to the possibility of reducing the dimension of the delayed variable, resulting in substantial savings of computational cost in stability analysis. We will first use a simple method to rewrite this system as coupled differential-difference equations without considering the reduction of dimensions. Introduce the new variable

$$y(t) = A_1 x(t). \tag{45}$$

Then the system can be written as

$$\dot{x}(t) = Ax(t) + y(t-r), \tag{46}$$
$$y(t) = A_1 x(t). \tag{47}$$

In this case, $B = I$, $C = A_1$, $D = 0$.

When $\mathrm{rank}(A_1) = k < n$, we will then try to decompose the system. The choice (41) results in $\mathcal{V} = \mathrm{Im}(A_1)$, then $d(\mathcal{V}) = k$. We will show that making the simple choice of $\mathcal{U} = \mathbb{R}^m$ results in an ad hoc method of reducing the dimension of the delayed variable y given in [4]. Indeed, with only y to be decomposed, we can write

$$y = Vy_1 + \hat{V}y_2, \tag{48}$$

where V has k columns and $\mathrm{Im}(V) = \mathrm{Im}(A_1)$, and the matrix $\begin{pmatrix} V & \hat{V} \end{pmatrix}$ is nonsingular. Let

$$\begin{pmatrix} V_i \\ \hat{V}_i \end{pmatrix} = \begin{pmatrix} V & \hat{V} \end{pmatrix}^{-1}. \tag{49}$$

Then

$$\hat{V}_i V = 0, \tag{50}$$

which also implies

$$\hat{V}_i A_1 = 0. \tag{51}$$

Then, the system becomes

$$\dot{x}(t) = Ax(t) + Vy_1(t-r) + \hat{V}y_2(t-r), \tag{52}$$
$$y_1(t) = V_i A_1 x(t), \tag{53}$$
$$y_2(t) = 0. \tag{54}$$

The above equations indicate that y_2 does not contribute to the dynamics of the system, and thus can be deleted from the model. This conclusion has been arrived in [4] by observing $A_1 = VP$ (here $P = V_i A_1$) and let $y_1(t) = Px(t)$. Therefore, such an ad hoc method is a special case of the method presented here.

It should be pointed out that in general, y_2 has its own dynamics that need to be evaluated separately. The above example is just a very special case where the dynamics of y_2 are trivial.

B. Unobservable difference equation

Suppose the pair (B, D) is unobservable. Let $\mathcal{V} = \mathcal{N}(B, D)$, then $\mathcal{V}$ is obviously D-invariant, and (20) is obviously satisfied because $B\mathcal{V} = \{0\}$. Therefore, Theorem 2 indicates that an A-invariant subspace $\mathcal{U}$ exists that satisfied (21). Indeed, in this case, (21) reduces to

$$\mathcal{U} \subset C^{-1}\mathcal{V}. \tag{55}$$

A valid choice is $\mathcal{U} = \{0\}$, in which case $x_2 = x$, and therefore, x is not decomposed, resulting in a system in which y_1 does not influence the dynamics of x and y_2.

Consider a system described by

$$\dot{y}(t) - D\dot{y}(t-r) = Ay(t) + Ey(t-r), \tag{56}$$

where $y(t) \in \mathbb{R}^n$, and

$$D = f_D g_D^T, \tag{57}$$
$$E = f_E g_E^T, \tag{58}$$

are rank 1 matrices, reflecting the common theme in practice that the delayed elements are of low dimensions. We assume the two column vectors f_D and f_E are linearly independent, and so are the vectors g_D and g_E. Let

$$x(t) = y(t) - Dy(t-r). \tag{59}$$

Then we can rewrite the system in the form of coupled differential-difference equations

$$\dot{x}(t) = Ax(t) + (AD+E)y(t-r), \tag{60}$$
$$y(t) = x(t) + Dy(t-r). \tag{61}$$

Compared with the standard form, we have $C = I$ and

$$B = AD + E = Af_D g_D^T + f_E g_E^T. \tag{62}$$

Select $V \in \mathbb{R}^{n\times(n-2)}$ with full column rank such that $g_D^T V = g_E^T V = 0$, and let $\mathcal{V} = \mathrm{Im}(V)$, then $\mathcal{V} = \mathcal{N}(B, D)$. Carry out the transformation

$$y = Vy_1 + \hat{V}y_2, \tag{63}$$

where $\hat{V}$ can be chosen to be $\begin{pmatrix} g_D & g_E \end{pmatrix}$. Let

$$\begin{pmatrix} V_i \\ \hat{V}_i \end{pmatrix} = \begin{pmatrix} V & \hat{V} \end{pmatrix}^{-1}, \tag{64}$$

then the transformed system can be written as

$$\dot{x}(t) = Ax(t) + (AD+E)\hat{V}y_2(t-r), \tag{65}$$
$$y_1(t) = V_i x(t) + V_i D\hat{V}y_2(t-r), \tag{66}$$
$$y_2(t) = \hat{V}_i x(t) + \hat{V}_i D\hat{V}y_2(t-r). \tag{67}$$

It can be seen that $y_1(t) \in \mathbb{R}^{n-2}$ does not affect the dynamics of x and y_2, and can be deleted from the system description. The remaining delayed variables are represented by $y_2(t) \in$

$\mathbb{R}^2$, which may have a much lower dimension than the original dimension n.

Although the dynamics of y_1 is trivial in this example (an algebraic expression of x and y_2), this is not always the case. In general, the dynamics of y_1 need to be evaluated separately.

C. Uncontrollable differential equation and unobservable differential equation

The readers should not have difficulty obtaining the conditions when the pair $(A, \mathrm{Im}(B))$ is uncontrollable or when the pair (C, A) is unobservable. These parallel results may be easily derived from Corollary 3, in a similar way Theorem 2 was used to derive results in the previous two subsections.

D. Finite number of invariant subspaces

When D or A have a finite number of invariant subspaces, a definite conclusion can be drawn whether it is possible to decompose the coupled differential-difference equations in the sense described in Section III. This is the case if each repeated eigenvalue of the matrix corresponds to a single Jordan block. For example, if D has only a finite number of invariant subspaces $\mathcal{V}_k, k = 1, 2, \ldots, K$, then, for each $\mathcal{V} = \mathcal{V}_k$, we may use Theorem 2 to check the existence of $\mathcal{U}$, thus exhaustively search whether any nontrivial S-invariant subspace $\mathcal{U}\oplus\mathcal{V}$ exists. This procedure will be illustrated in section VI.

V. Some additional observations

In this section, we would like to make a few additional observations.

First, we do not claim that this process provides two decoupled subsystems. In general, the output of the subsystem with variables x_2 and y_2 provides an input to the subsystem with variables x_1 and y_1. It is not always possible to decouple the system into two completely separate subsystems. Nonetheless, this structure is still valuable in analysis. For example, it is well known that the overall system is exponentially stable if and only if each subsystem is exponentially stable.

Second, if we follow the procedure described in Subsection IV-C, say, with the invariant subspaces of A. Then obviously $\mathcal{U} = \{0\}$ and $\mathcal{U} = \mathbb{R}^m$ are potential candidates. For $\mathcal{U} = \mathbb{R}^m$, we have seen in Subsection IV-A that $\mathcal{V}$ exists if the difference equation is uncontrollable. Similarly, for $\mathcal{U} = \{0\}$, we have seen in Subsection IV-B that $\mathcal{V}$ exists if the difference equation is unobservable. It is interesting to ask the opposite question: if we choose $\mathcal{U} = \{0\}$ or $\mathcal{U} = \mathbb{R}^m$, what are the conditions needed for $\mathcal{V}$ to exist? The following theorem provides a definite answer.

Theorem 5. *For $\mathcal{U} = \mathbb{R}^m$, there exists a subspace $\mathcal{V} \subset \mathbb{R}^n$, $\mathcal{V} \neq \mathbb{R}^n$, such that $\mathcal{U}\oplus\mathcal{V}$ is S-invariant if and only if the pair $(D, \mathrm{Im}(C))$ is uncontrollable. For $\mathcal{U} = \{0\}$, there exists a subspace $\mathcal{V} \subset \mathbb{R}^n$, $\mathcal{V} \neq \{0\}$, such that $\mathcal{U}\oplus\mathcal{V}$ is S-invariant if and only if the pair (B, D) is unobservable.*

Proof. For $\mathcal{U} = \mathbb{R}^m$, if $(D, \mathrm{Im}(C))$ is controllable, then the equation (23) in Corollary 3 requires $\mathcal{V} = \mathbb{R}^n$, and therefore, the only S-invariant subspace is $\mathbb{R}^{m+n}$, which is trivial. On the other hand, if $(D, \mathrm{Im}(C))$ is uncontrollable, Subsection IV-A has already shown that $\mathcal{V} = \langle D \mid \mathrm{Im}(C)\rangle, \mathcal{V} \neq \mathbb{R}^n$, makes $\mathcal{U}\oplus\mathcal{V}$ a nontrivial S-invariant subspace. This proved the first part.

For $\mathcal{U} = \{0\}$, the equation (23) in Corollary 3 requires $\mathcal{V} \subset \mathrm{Ker}(B)$. Because $\mathcal{V}$ is D-invariant, this means $\mathcal{V} \subset \mathcal{N}(B, D)$. If (B, D) is observable, then $\mathcal{V} = \{0\}$, and $\mathcal{U}\oplus\mathcal{V}$ is trivial. If (B, D) is unobservable, then Subsection IV-B has already shown that $\mathcal{V} = \mathcal{N}(B, D) \neq 0$ makes $\mathcal{U}\oplus\mathcal{V}$ a nontrivial S-invariant subspace. ■

Third, the decomposition of coupled differential-difference equations is independent of the value of delay r or whether the delay is constant or depends on time. While the main motivation here is the stability analysis of systems with time-invariant delay using a complete type of Lyapunov-Krasovskii functional, it may certainly be applied to other analysis and systems with time-varying delay.

VI. Illustrative example

In this section, we will illustrate the procedure of discovering all the possible decomposition of coupled differential-difference equations when the matrix D has a finite number of invariant subspaces. Consider the system:

$$\dot{x}(t) = Ax(t) + By(t-r), \tag{68}$$
$$y(t) = Cx(t) + Dy(t-r), \tag{69}$$

where

$$A = \begin{pmatrix} 2 & 14 & -7 & 1 \\ 6 & 3 & 9 & -8 \\ 27 & -35 & 48 & -28 \\ \frac{116}{3} & -51 & \frac{182}{3} & -34 \end{pmatrix}, B = \begin{pmatrix} \frac{61}{3} & \frac{-43}{3} & \frac{40}{3} \\ 19 & -14 & 12 \\ \frac{53}{3} & \frac{-26}{3} & \frac{11}{3} \\ \frac{62}{3} & \frac{-20}{3} & \frac{-1}{3} \end{pmatrix},$$
$$C = \begin{pmatrix} 34 & -57 & 82 & -46 \\ 25 & -40 & 66 & -37 \\ 22 & -37 & 54 & -30 \end{pmatrix}, D = \begin{pmatrix} \frac{38}{3} & \frac{-26}{3} & \frac{20}{3} \\ \frac{44}{3} & \frac{-26}{3} & \frac{20}{3} \\ 7 & -5 & 5 \end{pmatrix},$$

and let

$$S = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{70}$$

The matrix D has a simple eigenvalue $\lambda_1 = 5$ and a repeated eigenvalue $\lambda_2 = \lambda_3 = 2$. The eigenvector corresponding to λ_1 is $V_1 = \left(\frac{100}{27}, \frac{140}{27}, \frac{67}{27}\right)^T$. Corresponding to the repeated eigenvalue λ_2, there is one eigenvector $V_2 = \left(\frac{-4}{9}, \frac{-8}{9}, \frac{-4}{9}\right)^T$, and one generalized eigenvector $V_3 = \left(\frac{-73}{27}, \frac{-140}{27}, \frac{-67}{27}\right)^T$. Let $\mathcal{V}_i = \mathrm{span}\{V_i\}, i = 1, 2, 3$. Then the invariant subspaces of D are $\{0\}, \mathcal{V}_1, \mathcal{V}_2, \mathcal{V}_1+\mathcal{V}_2, \mathcal{V}_2+\mathcal{V}_3$, and $\mathbb{R}^3$. Invoking Theorem 2, it can be checked that the relation (20) is satisfied by $\mathcal{V} = \{0\}, \mathcal{V}_2, \mathcal{V}_2+\mathcal{V}_3$, and $\mathbb{R}^3$, and not satisfied by the other invariant subspaces of D. Because $(A, \mathrm{span}\{B\})$ and $(D, \mathrm{span}\{C\})$ are controllable, and (C, A) and (B, D) are observable, $\mathcal{V} = \{0\}$ or $\mathcal{V} = \mathbb{R}^3$ does not lead to nontrivial invariant subspace of S according to Theorem 5.

In the following, we will obtain decomposition of the system by applying Theorem 2 to the remaining two choices of $\mathcal{V}$, viz, $\mathcal{V} = \mathcal{V}_2$ and $\mathcal{V} = \mathcal{V}_2+\mathcal{V}_3$.

A. 1-dimensional invariant subspace of D

Let $\mathcal{V} = \mathcal{V}_2$, then, it can be calculated that

$$C\langle A|B\mathcal{V}\rangle = \mathcal{V} = \mathrm{span}\{V\}, \quad (71)$$

where

$$V = \begin{pmatrix} \frac{-4}{9} \\ \frac{-8}{9} \\ \frac{-4}{9} \end{pmatrix}. \quad (72)$$

Thus, the equation (20) in Theorem 2 is satisfied. We now proceed to find $\mathcal{U}$ to satisfy (21). We can calculate,

$$\langle A|B\mathcal{V}\rangle = \mathrm{span}\begin{pmatrix} \frac{-20}{9} & \frac{-124}{9} \\ \frac{-4}{3} & \frac{-76}{9} \\ \frac{-16}{9} & \frac{-104}{9} \\ \frac{-28}{9} & 20 \end{pmatrix}, \quad (73)$$

$$C^{-1}\mathcal{V} = \mathrm{span}\begin{pmatrix} \frac{1000}{1249} & \frac{68}{351} \\ \frac{400}{1249} & \frac{-4}{351} \\ \frac{200}{1249} & \frac{-80}{351} \\ \frac{600}{1249} & \frac{-28}{117} \end{pmatrix}. \quad (74)$$

It is easily verified that $\langle A|B\mathcal{V}\rangle = C^{-1}\mathcal{V}_2$. Thus, for the given 1-dimensional invariant subspace $\mathcal{V}$, the corresponding $\mathcal{U} = \mathrm{span}(U)$ is the unique 2-dimensional subspace $\mathcal{U} = \mathrm{span}\{U\}$, where

$$U = \begin{pmatrix} \frac{-20}{9} & \frac{-124}{9} \\ \frac{-4}{3} & \frac{-76}{9} \\ \frac{-16}{9} & \frac{-104}{9} \\ \frac{-28}{9} & 20 \end{pmatrix}. \quad (75)$$

To decompose the coupled differential-difference equations (68-69), introduce a variable transformation:

$$x = Ux_1 + \hat{U}x_2, \quad (76)$$
$$y = Vy_1 + \hat{V}y_2, \quad (77)$$

where

$$\hat{U} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \\ 3 & 2 \end{pmatrix}, \hat{V} = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ 3 & 1 \end{pmatrix}, \quad (78)$$

which are chosen such that $\begin{pmatrix} U & \hat{U} \end{pmatrix}$ and $\begin{pmatrix} V & \hat{V} \end{pmatrix}$ are nonsingular. The transformed equations become:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \tilde{A}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \tilde{B}\begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \quad (79)$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \tilde{C}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \tilde{D}\begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \quad (80)$$

where

$$\tilde{A} = \begin{pmatrix} U & \hat{U} \end{pmatrix}^{-1} A \begin{pmatrix} U & \hat{U} \end{pmatrix} = \begin{pmatrix} 0 & 1 & \frac{-945}{8} & \frac{10431}{8} \\ 1 & 6 & \frac{75}{4} & \frac{-843}{4} \\ 0 & 0 & \frac{49}{6} & \frac{-17}{6} \\ 0 & 0 & \frac{-1}{6} & \frac{29}{6} \end{pmatrix}, \quad (81)$$

$$\tilde{B} = \begin{pmatrix} U & \hat{U} \end{pmatrix}^{-1} B \begin{pmatrix} V & \hat{V} \end{pmatrix} = \begin{pmatrix} 1 & \frac{-5073}{8} & \frac{5721}{16} \\ 0 & \frac{405}{4} & \frac{-117}{2} \\ 0 & \frac{41}{2} & \frac{41}{4} \\ 0 & \frac{-3}{2} & \frac{3}{4} \end{pmatrix}, \quad (82)$$

$$\tilde{C} = \begin{pmatrix} V & \hat{V} \end{pmatrix}^{-1} C \begin{pmatrix} U & \hat{U} \end{pmatrix} = \begin{pmatrix} 5 & 33 & 144 & \frac{6525}{4} \\ 0 & 0 & 20 & 208 \\ 0 & 0 & 24 & 244 \end{pmatrix}, \quad (83)$$

$$\tilde{D} = \begin{pmatrix} V & \hat{V} \end{pmatrix}^{-1} D \begin{pmatrix} V & \hat{V} \end{pmatrix} = \begin{pmatrix} 2 & \frac{327}{4} & \frac{-363}{8} \\ 0 & \frac{35}{3} & \frac{-29}{6} \\ 0 & \frac{40}{3} & \frac{-14}{3} \end{pmatrix}. \quad (84)$$

From the above expressions of $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, it is easily seen that the variables x_2 and y_2 are independent of x_1 and y_1, and thus a decomposition has been achieved.

B. 2-dimensional invariant subspace of D

Now let $\mathcal{V} = \mathcal{V}_2 + \mathcal{V}_3$ instead. Then,

$$C\langle A|B\mathcal{V}\rangle = \mathcal{V} = \mathrm{span}\{V\}, \quad (85)$$

where

$$V = \begin{pmatrix} \frac{-4}{9} & \frac{-73}{27} \\ \frac{-8}{9} & \frac{-140}{27} \\ \frac{-4}{9} & \frac{-67}{27} \end{pmatrix}, \quad (86)$$

and, the equation (20) in Theorem 2 is satisfied. To find $\mathcal{U}$ to satisfy (21). We calculate,

$$\langle A|B\mathcal{V}\rangle = \mathrm{span}\begin{pmatrix} \frac{-20}{9} & \frac{-371}{27} \\ \frac{-4}{3} & \frac{-77}{9} \\ \frac{-16}{9} & \frac{-322}{27} \\ \frac{-28}{9} & \frac{-553}{27} \end{pmatrix}, \quad (87)$$

$$C^{-1}\mathcal{V} = \mathrm{span}\begin{pmatrix} \frac{1000}{1249} & \frac{68}{351} & \frac{1142}{1345} \\ \frac{400}{1249} & \frac{-4}{351} & \frac{-175}{1053} \\ \frac{200}{1249} & \frac{-80}{351} & \frac{-354}{341} \\ \frac{600}{1249} & \frac{-28}{117} & \frac{-781}{815} \end{pmatrix}. \quad (88)$$

It can be verified that $\langle A|B\mathcal{V}\rangle$ is a subspace of $C^{-1}\mathcal{V}$. The corresponding $\mathcal{U}$ must be A-invariant. Obviously, $\langle A|B\mathcal{V}\rangle$ is A-invariant. However, $C^{-1}\mathcal{V}$ is not. Therefore, the only choice of $\mathcal{U}$ that is A-invariant and satisfies (21) is $\mathcal{U} = \langle A|B\mathcal{V}\rangle = \mathrm{span}(U)$, where

$$U = \begin{pmatrix} \frac{-20}{9} & \frac{-371}{27} \\ \frac{-4}{3} & \frac{-77}{9} \\ \frac{-16}{9} & \frac{-322}{27} \\ \frac{-28}{9} & \frac{-553}{27}. \end{pmatrix}. \quad (89)$$

A variable transformation to accomplish decomposition is:

$$x = Ux_1 + \hat{U}x_2, \quad (90)$$
$$y = Vy_1 + \hat{V}y_2, \quad (91)$$

where

$$\hat{U} = \begin{pmatrix} 1 & 2 \\ 2 & 3 \\ 3 & 5 \\ 3 & 2 \end{pmatrix}, \hat{V} = \begin{pmatrix} 3 \\ 5 \\ 1 \end{pmatrix}. \tag{92}$$

The transformed equations become:

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \tilde{A} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \tilde{B} \begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \tag{93}$$

$$\begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix} = \tilde{C} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \tilde{D} \begin{pmatrix} y_1(t-r) \\ y_2(t-r) \end{pmatrix}, \tag{94}$$

where

$$\tilde{A} = \left(U \ \ \hat{U}\right)^{-1} A \left(U \ \ \hat{U}\right) = \begin{pmatrix} \frac{8}{3} & \frac{623}{36} & \frac{-545}{8} & \frac{5935}{8} \\ \frac{4}{7} & \frac{10}{3} & \frac{75}{7} & \frac{-843}{7} \\ 0 & 0 & \frac{49}{6} & \frac{-17}{6} \\ 0 & 0 & \frac{-1}{6} & \frac{29}{6} \end{pmatrix}, \tag{95}$$

$$\tilde{B} = \left(U \ \ \hat{U}\right)^{-1} B \left(V \ \ \hat{V}\right) = \begin{pmatrix} 1 & 0 & \frac{3225}{16} \\ 0 & 1 & \frac{-234}{7} \\ 0 & 0 & \frac{-41}{4} \\ 0 & 0 & \frac{3}{4} \end{pmatrix}, \tag{96}$$

$$\tilde{C} = \left(V \ \ \hat{V}\right)^{-1} C \left(U \ \ \hat{U}\right) = \begin{pmatrix} 5 & \frac{413}{12} & 924 & \frac{38973}{4} \\ 0 & 0 & -180 & -1872 \\ 0 & 0 & -16 & -172 \end{pmatrix}, \tag{97}$$

$$\tilde{D} = \left(V \ \ \hat{V}\right)^{-1} D \left(V \ \ \hat{V}\right) = \begin{pmatrix} 2 & 1 & \frac{-1871}{8} \\ 0 & 2 & \frac{87}{2} \\ 0 & 0 & 5 \end{pmatrix}. \tag{98}$$

Again, from the above expressions of $\tilde{A}, \tilde{B}, \tilde{C}, \tilde{D}$, it is easy to see that, the variables x_2 and y_2 are independent of x_1 and y_1, thus a decomposition has been achieved.

VII. Further remarks

In this section, we will make a few additional remarks.

First, we do not claim that the procedure described in this paper can find a transformation to decompose the system whenever such one exists. However, such cases are rare. Indeed, the only case that our procedure fails is when $(A, \mathrm{Im}(B))$ and $(D, \mathrm{Im}(C))$ are both controllable, (C, A) and (B, D) are both observable, and both A and D have repeated eigenvalues that correspond to more than one Jordan blocks. In practice, one may start with checking the controllability and observability and see if the procedure described by Subsections IV-A, IV-B or IV-C would apply. Then check if either D or A has a finite number of invariant subspaces, and follow the procedure illustrated in Section VI if it is the case.

Second, if the system has inputs, for example,

$$\dot{x}(t) = Ax(t) + By(t-r) + Eu(t), \tag{99}$$

$$y(t) = Cx(t) + Dy(t-r) + Fu(t) \tag{100}$$

Then, the transformation does not consider the input terms, and the resulting system does not decompose the input. This is similar to the case discussed in [8].

Third, in addition to coupled differential-difference equations, this decomposition may be applied to other context. For example, for two sets of coupled difference equations

$$x(t) = Ax(t-r_1) + By(t-r_2), \tag{101}$$

$$y(t) = Cx(t-r_1) + Dy(t-r_2), \tag{102}$$

where one or both delays r_1 and r_2 may be time-varying. The procedure described in this article can obviously be used to decompose the system without change.

VIII. Conclusions and future work

Invariant subspace that is restricted to be an external direct sum of two subspaces is directly related to the decomposition of coupled differential-difference equations. Such invariant subspaces possess a number of interesting properties that may facilitate their discovery. A number of known simple ad hoc methods of reducing the dimensions of coupled differential-difference equations can be considered as special cases of such invariant subspace problem.

It is interesting to extend this method to the case of multiple delays. The significance of this problem and an ad hoc method of reducing the dimensions of delayed variables in writting coupled differential-difference equations with multiple delays are given in [8].

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