Some insights into the migration of double imaginary roots under small deviation of two parameters

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Some geometric insights into the migration of double imaginary roots under small deviation of two parameters

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Abstract

This paper studies the migration of double imaginary characteristic roots of the system’s characteristic equation when two parameters are subjected to small deviations. The proposed approach covers a wide range of models. Under certain assumptions, it was found that the local stability crossing curve has a cusp at the point that corresponds to the double root, and it divides the neighborhood of this point into an S-sector and a G-sector. When the parameters move into the G-sector, one of the roots moves to the right half-plane, and the other moves to the left half-plane. When the parameters move into the S-sector, both roots move either to the left half-plane or the right half-plane function of the sign of some value that depends on the characteristic function and its derivatives up to the third order.

Key words: Characteristic roots; Spectral analysis; Time delay; Stability analysis; Stability criteria.

1 Introduction

Control systems often depend on parameters, such that their characteristic equation may be written as

\[ q(s, p) = 0, \]  

where \( s \) is the Laplace variable and \( p \in \mathbb{R}^n \) is a vector of \( n \) parameters. We may have parameters due to internal dynamics. For instance, modeling in physical, biological or social sciences sometimes requires taking into account the time delays inherent in the phenomena. Depending on the model complexity, but also on how much information is known, we may choose a model with continuous constant delays, or a model with distributed delays (see Cushing, 1977; MacDonald, 1989). For instance, in the case of a time-delay system with two constant delays, the characteristic equation is written of the form

\[ q_1(s, \tau_1, \tau_2) = r_0(s) + r_1(s)e^{-\tau_1 s} + r_2(s)e^{-\tau_2 s}, \]  

where \( r_k(s), k = 0, 1, 2 \) are polynomials of \( s \) with real coefficients, and the delays \( \tau_1, \tau_2 \) are the two parameters.

Also common is the case when \( p \) contains controller parameters. Classical example are PI, PD and PID controllers, where the continuous time domain controller is expressed in the Laplace domain as

\[ q_2(s) = K_P \left(1 + \frac{1}{T_i s}\right), \]  

\[ q_3(s) = K_P \left(1 + T_d s\right), \]  

\[ q_4(s) = K_P \left(1 + \frac{1}{T_i s} + T_d s\right), \]  

respectively, where \( K_P \) is the proportional gain, \( T_i \) the integral time, and \( T_d \) the derivative time. Furthermore, in many practical applications, a time delay of the process model \( \tau_m \) may be involved (see O’Dwyer, 2006; Morarescu, Mendez-Barrios, Niculescu & Gu, 2011). These include proportional plus delay (6), integrator plus delay model (7), first order lag plus delay (8), first
order lag plus integral plus delay (9) expressed below:

\[ q_6(s) = K_m (1 + e^{-sT_m}) \]  
\[ q_0(s) = \frac{K_m e^{-sT}}{s} \]  
\[ q_7(s) = \frac{K_m e^{-sT}}{1 + sT_m} \]  
\[ q_8(s) = \frac{K_m e^{-sT_m}}{s(1 + sT_m)} \]

If in equation (6) there are two different proportional gains, then we obtain the model of a proportional retarded controller:

\[ q_0(s) = K_p + K_r e^{-sT_m}. \]  

Furthermore, Villafuerte, Mondié & Garrido (2013) showed that proportional retarded controller outperforms a PD controller on an experimental DC-servomotor setup. Obviously, any control among (3) to (10) results in a characteristic equation that depends on the control parameters.

Many studies have been conducted on the stability of systems that depend on parameters. For example, for systems with a single delay as the parameter, methods of identifying all the stable delay intervals are given in Lee & Hsu (1969) and Walton & Marshall (1987). For systems with two parameters, a rich collection of stability charts (the parameter regions where the system is stable) for time delay systems are presented in Stépán (1989). For systems with two delays as the parameters, a geometric approach is introduced in Gu, Niculescu & Chen (2005). The analysis is based on the continuity of the characteristic roots as functions of parameters (which needs to be carefully evaluated in the case of time delay systems of neutral type (see Gu, 2012; Michiels & Niculescu, 2007)), and consists of identifying the parameters that correspond to imaginary characteristic roots and judging the direction of crossing of these roots as parameters change. Such an analysis is often known as D-subdivision method (Gryazina, Polyak & Tremba, 2008).

Challenges due to non-differentiability arise when the imaginary roots concerned are multiple roots. Such problems have traditionally been solved using Puiseux series (Kato, 1980; Knopp, 1996), see, for example, Chen, Fu, Niculescu & Guan (2010a), Chen, Fu, Niculescu & Guan (2010b) and Li, Niculescu, Cela, Wang & Cai (2013) for systems with one parameter.

In this paper, we study systems with two parameters, and present a method to analyze the migration of roots in a neighborhood of the parameter corresponding to a double imaginary characteristic root. The method of analysis uses traditional complex analysis, and does not require puiseux series. A preliminary version of this paper, which is restricted to the case of two point-wise delays as the parameters, was presented in Gu, Irofti, Boussaada & Niculescu (2015). In Sections 2–4 we extend and generalize this method to a wide range of systems, as mentioned above, that can generally be written of the form of characteristic equation (1). Additionally, we illustrate how to apply the algebraic criterion given in Section 5 to three examples of different nature (parameter-dependent polynomials, time-delay systems, and distributed delays systems), illustrated in Section 6; moreover, degenerate cases are identified and discussed. Brief concluding remarks can be found in Section 7.

2 Problem statement

Consider a system with the characteristic equation of the form (1). For \( p_0 = (p_{10}, p_{20}) \), we assume that the function \( q(s, p_0) \) has a double root on the imaginary axis, \( s = s_0 = i\omega_0 \). In other words, we assume

\[ q(s_0, p_0) = 0, \]  
\[ \frac{\partial q}{\partial s} \bigg|_{s=s_0, p=p_0} = 0. \]  

We further assume that \( s_0 \) is not a third order root, i.e.

\[ \frac{\partial^2 q}{\partial s^2} \bigg|_{s=s_0, p=p_0} \neq 0. \]

We assume that \( q(s, p) \) is analytic with respect to \( s \), and continuously differentiable with respect to \((s, p)\) up to a third order. We make the following additional non-degeneracy assumption:

\[ D = \det \left( \begin{array}{cc} \text{Re} \left( \frac{\partial q}{\partial p_1} \right) & \text{Re} \left( \frac{\partial q}{\partial p_2} \right) \\ \text{Im} \left( \frac{\partial q}{\partial p_1} \right) & \text{Im} \left( \frac{\partial q}{\partial p_2} \right) \end{array} \right) \bigg|_{s=s_0, p_1=p_{10}, p_2=p_{20}} \neq 0, \]  

where \( \text{Re}(\cdot) \) and \( \text{Im}(\cdot) \) denote the real and imaginary part of a complex number, respectively. Equations (11)-(14) will be the standing assumptions in the remaining part of this paper.

Note that \( D \) in (14) may also be written as

\[ D = \text{Im} \left( \frac{\partial q^*}{\partial p_1} \cdot \frac{\partial q}{\partial p_2} \right) \bigg|_{s=s_0, p_1=p_{10}, p_2=p_{20}}, \]

where \((\cdot)^*\) denotes the complex conjugate of a complex number.

Obviously, if (1) satisfies (11) and (12), then (13) and (14) represents the “least degenerate” case. Therefore, we will refer to (13) and (14) as the least degeneracy assumptions. In view of implicit function theorem, a consequence of inequality (14), which is part of the non-
the point \( p \)ing curves to a neighborhood of \( T \) region. We will also denote parameter space into regions, such that the number of curve. Roughly speaking, it is a curve that divides the axis. Therefore, the set \( \mathcal{N}_s(x_0) = \{ x \mid |x - x_0| < \varepsilon \} \).

Then the above can be more precisely stated as follows.

**Proposition 1** There exists an \( \varepsilon > 0 \) and a sufficiently small \( \delta > 0 \) such that for all \( s \in \mathcal{N}_s(s_0) \), we may define \( p_1(s) \) and \( p_2(s) \) as the unique solution of (1) with \( (p_1(s), p_2(s)) \in \mathcal{N}_e(p_{10}, p_{20}) \). The functions so defined are differentiable up to the third order.

It should be pointed out that in general, for \( s \in \mathcal{N}_s(s_0) \), characteristic equation (1) may have other solutions outside of \( \mathcal{N}_e(p_{10}, p_{20}) \).

Recall that stability crossing curves are defined in Gu et al. (2005) as the set of all points \( (p_1, p_2) \in \mathbb{R}^2_+ \) such that \( q(s) \) has at least one zero on the imaginary axis. Therefore, the set

\[
\mathcal{T}_{(\omega_0, p_{10}, p_{20})} = \{(p_1(\omega), p_2(\omega)) \in \mathcal{N}_e(p_{10}, p_{20}) \mid \omega \in \mathcal{N}_e(i\omega_0) \},
\]

which is a curve in the \( p_1-p_2 \) space that passes through the point \( (p_{10}, p_{20}) \), is the restriction of stability crossing curves to a neighborhood of \( (p_{10}, p_{20}) \). Therefore, \( \mathcal{T}_{(\omega_0, p_{10}, p_{20})} \) will be known as the local stability crossing curve. Roughly speaking, it is a curve that divides the parameter space into regions, such that the number of characteristic roots on the right half complex plane remain constant as the parameters vary within each such region. We will also denote

\[
\mathcal{T}_{(\omega_0, p_{10}, p_{20})}^+ = \{(p_1(\omega), p_2(\omega)) \in \mathcal{N}_e(p_{10}, p_{20}) \mid \omega \in \mathcal{N}_e(i\omega_0), \omega > \omega_0 \}
\]

and

\[
\mathcal{T}_{(\omega_0, p_{10}, p_{20})}^- = \{(p_1(\omega), p_2(\omega)) \in \mathcal{N}_e(p_{10}, p_{20}) \mid \omega \in \mathcal{N}_e(i\omega_0), \omega < \omega_0 \}.
\]

The curves \( \mathcal{T}_{(\omega_0, p_{10}, p_{20})}^+ \) and \( \mathcal{T}_{(\omega_0, p_{10}, p_{20})}^- \) will be known as the positive and negative local stability crossing curves, respectively.

The purpose of this paper is to study how the two characteristic roots migrate as \( (p_1, p_2) \) varies in a small neighborhood of \( (p_{10}, p_{20}) \) under the least degeneracy assumptions.

3 Cusp and local bijection

Let

\[
s = s_0 + \varepsilon e^{i\theta}.
\]

Then \( u \) and \( \theta \) parameterize a neighborhood of \( s_0 \), and \( p_1 \) and \( p_2 \) can be considered as functions of \( u \) and \( \theta \). For the sake of convenience, write

\[
\gamma = e^{i\theta} = \frac{\partial s}{\partial u}.
\]

We first fix the angular variable \( \theta \), i.e., fix \( \gamma \), and calculate the derivatives of \( p_1 \) and \( p_2 \) with respect to the radial variable \( u \). This can be easily achieved by differentiating (1), yielding

\[
\frac{\partial q}{\partial p_1} \frac{\partial p_1}{\partial u} + \frac{\partial q}{\partial p_2} \frac{\partial p_2}{\partial u} + \frac{\partial q}{\partial s} \gamma = 0.
\]

Setting \( u = 0 \) and using (12) in (18), we obtain

\[
\left( \begin{array}{c}
\text{Re} \left( \frac{\partial q}{\partial p_1} \right) \\
\text{Im} \left( \frac{\partial q}{\partial p_1} \right)
\end{array} \right) \left( \begin{array}{c}
\text{Re} \left( \frac{\partial q}{\partial p_2} \right) \\
\text{Im} \left( \frac{\partial q}{\partial p_2} \right)
\end{array} \right)_{s=s_0, p_1=p_{10}, p_2=p_{20}} = 0,
\]

from which we conclude

\[
\frac{\partial p_1}{\partial s} = 0,
\]

in view of (14). Equation (19) has two important implications.

First, if we set \( \gamma = i \), the equation (19) indicates that the local stability crossing curve \( \mathcal{T}_{(\omega_0, p_{10}, p_{20})} \) may have a cusp at \( (p_{10}, p_{20}) \) (see Guggenheimer, 1977). Indeed, as will be confirmed by considering the second-order derivative in the next section, \( \mathcal{T}_{(\omega_0, p_{10}, p_{20})} \) partitions a sufficiently small neighborhood of \( (p_{10}, p_{20}) \) into a great sector (or G-sector) and a small sector \(^1\) (or S-sector) as shown in Figure 1. We will investigate how the double roots at \( i\omega_0 \) migrate as \( (p_1, p_2) \) moves from \( (p_{10}, p_{20}) \) to the G-sector or the S-sector.

To obtain the second implication, we first show the following.

**Lemma 2** Consider \( s_a \in \mathcal{N}_e^2(s_0), \delta > 0 \) sufficiently small, and let \( p_{1a} = p_1(s_a), p_{2a} = p_2(s_a) \) as defined in

\(^1\) We have used the word “small” in a sense analogous to “small solution”: a small sector is contained by a sector with straight sides with arbitrarily small angle when the neighborhood is sufficiently small.
Fig. 1. G-sector and S-sector.

**Proposition 1.** Then

\[
\frac{\partial}{\partial s} q(s, p_{10}, p_{2a}) \bigg|_{s = s_0} \neq 0. \tag{20}
\]

**Proof.** Let

\[
s_a = s_0 + u \gamma, \quad |\gamma| = 1,
\]

then,

\[
\frac{\partial q}{\partial s} \bigg|_{s = s_a} = \frac{\partial q}{\partial s} \bigg|_{s = s_0} + \frac{\partial^2 q}{\partial s^2} \bigg|_{s = s_0} \gamma u
\]

\[+
\frac{\partial^2 q}{\partial s \partial p_1} \bigg|_{s = s_0} \frac{\partial p_1}{\partial u} \bigg|_{u = 0} + \frac{\partial^2 q}{\partial s \partial p_2} \bigg|_{s = s_0} \frac{\partial p_2}{\partial u} \bigg|_{u = 0} + o(u)
\]

\[= 0 + \frac{\partial^2 q}{\partial s^2} \bigg|_{s = s_0} \gamma u + 0 + 0 + o(u),
\]

from which we may conclude (20) in view of (13).

The implicit function theorem allows us to conclude the following from Lemma 2.

**Proposition 3** Let \( s_a, p_{10}, \) and \( p_{2a} \) be defined as in Lemma 2. Then there exists a sufficiently small neighborhood of \( (p_{10}, p_{2a}) \) such that the equation (1) defines a unique function \( s(p_1, p_2) \) with the function value restricted in a small neighborhood of \( s_a \).

The second implication of the equation (19) may be stated as the following corollary, which is a consequence of Propositions 1 and 3.

**Corollary 4** Let \( s_a, p_{10}, \) and \( p_{2a} \) be defined as in Lemma 2. Then equation (1) defines a bijection between \( s \) in a small neighborhood of \( s_a \) and \( (p_1, p_2) \) in a small neighborhood of \( (p_{10}, p_{2a}) \).

Obviously, the small neighborhoods referred in Proposition 3 and Corollary 4 above should not include \( s_0 \) and \((p_{10}, p_{20})\) in view of condition (12). Moreover, in view of continuity of solutions of (1) with respect to the parameters \((p_1, p_2)\), Corollary 4 may be equivalently stated as follows.

**Corollary 5** For all \((p_1, p_2) \in \mathcal{N}_\varepsilon^2(p_{10}, p_{20}) \) with \( \varepsilon > 0 \) sufficiently small, the characteristic equation (1) has exactly two simple roots in a small neighborhood of \( s_0 \).

### 4 Mapping in a neighborhood of double root

In this section, it will be shown that we can very clearly describe the mapping between \( s \) and \((p_1, p_2)\) in the neighborhood of \( s_0 \) based on the second order derivative when \( s = s_0 \) is restricted to one quadrant. From this description, we may obtain the information on how the double root migrates as \((p_1, p_2)\) moves from \((p_{10}, p_{20})\) to the G-sector or the S-sector in Figure 1 according to the sign of \( D \), and whether the negative local stability crossing curve \( T_{(\omega_0, p_{10}, p_{20})} \) is on the clockwise side or on the counterclockwise side of \( T_{(\omega_0, p_{10}, p_{20})} \) in the S-sector.

Taking derivative of (18) with respect to the radial variable \( u \), we obtain

\[
\frac{\partial^2 q}{\partial p_1^2} \left( \frac{\partial p_1}{\partial u} \right)^2 + 2 \frac{\partial^2 q}{\partial p_1 \partial p_2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} + 2 \frac{\partial^2 q}{\partial p_2^2} \left( \frac{\partial p_2}{\partial u} \right)^2 + 2 \frac{\partial q}{\partial p_1} \frac{\partial q}{\partial p_2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} + 2 \frac{\partial q}{\partial p_1} \frac{\partial q}{\partial p_2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} + 2 \frac{\partial q}{\partial p_1} \frac{\partial q}{\partial p_2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} + 2 \frac{\partial q}{\partial p_1} \frac{\partial q}{\partial p_2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} = 0. \tag{21}
\]

Setting \( u = 0 \) and applying (19) in (21), we arrive at

\[
\left[ \frac{\partial q}{\partial p_1} \frac{\partial^2 p_1}{\partial u^2} + \frac{\partial q}{\partial p_2} \frac{\partial^2 p_2}{\partial u^2} + \frac{\partial^2 q}{\partial s^2} \gamma^2 \right]_{s = s_0} = 0.
\]

The above may be solved for \( \frac{\partial^2 p_1}{\partial u^2} \) and \( \frac{\partial^2 p_2}{\partial u^2} \) to obtain,

\[
\left( \begin{array}{c}
\frac{\partial^2 p_1}{\partial u^2} \\
\frac{\partial^2 p_2}{\partial u^2}
\end{array} \right)_{s = s_0} = \left[ \begin{array}{cc}
\Re \frac{\partial q}{\partial p_1} & \Re \frac{\partial q}{\partial p_2} \\
\Im \frac{\partial q}{\partial p_1} & \Im \frac{\partial q}{\partial p_2}
\end{array} \right]^{-1} \left[ \begin{array}{c}
\Re \left( \frac{\partial^2 q}{\partial s^2} \gamma^2 \right) \\
\Im \left( \frac{\partial^2 q}{\partial s^2} \gamma^2 \right)
\end{array} \right]_{s = s_0}.
\]

(22)
which may also be written in a complex form

\[
\begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix}
\begin{bmatrix}
s = s_0 \\
p_1 = p_{10} \\
p_2 = p_{20}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix}^{-1}
\begin{bmatrix}
ds = s_0 \\
p_1 = p_{10} \\
p_2 = p_{20}
\end{bmatrix}
\begin{bmatrix}
r_{10} \\
r_{20}
\end{bmatrix}
\begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix}^{-1}
\begin{bmatrix}
r_{10} \\
r_{20}
\end{bmatrix}.
\]

(23)

In view of (19), the tangent of the curve describing \((p_1, p_2)\) as a function of \((p_{10}, p_{20})\) is determined by the second order derivative given in (22) or (23).

Before proceeding further, it is helpful to recall the following well known fact. It can be found in various elementary books that deal with geometry, see for example Gonzalez & Stuart (2008).

Lemma 6 Let \(x^{(0)} \in \mathbb{R}^2\) and \(M \in \mathbb{R}^{2 \times 2}\) be fixed. For any \(x \in \mathbb{R}^2\), let \(\theta\) be the angle to rotate \(x^{(0)}\) to the direction of \(x\) in the counterclockwise direction. Let \(\phi(\theta)\) be the angle to rotate \(Mx^{(0)}\) to the direction of \(Mx\) in the counterclockwise direction if \(\det(M) > 0\), and in the clockwise direction if \(\det(M) < 0\). Then the function \(\phi(\theta)\) satisfies the following:

i) \(\phi(\theta)\) is a continuous and increasing function of \(\theta\)

ii) \(0 < \phi(\theta) < \pi\) if and only if \(0 < \theta < \pi\).

We now make the following two observations about the second order derivative expression (22).

First, set \(\gamma = i\) and \(\gamma = -i\), the expression determines the tangent of \(T_{(\omega_0, p_{10}, p_{20})}\) as \(\omega \to \omega_0\) from each side.

As \(\begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix}^T\) given in (22) for \(\gamma = i\) and \(-i\) have the same value, \(T_{(\omega_0, p_{10}, p_{20})}^-\) and \(T_{(\omega_0, p_{10}, p_{20})}^+\) (\(A'C'\) and \(C'E'\) in Figure 1) are tangent to each other at the point \((p_{10}, p_{20})\), thus forming a cusp.

Second, as \(\gamma\) rotates through a 90° angle in a counterclockwise direction, \(\frac{\partial^2 p_i}{\partial u^2}\) rotates through a 180° angle in the same direction; and \(\begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix}^T\) given in (22) also rotates through a 180° angle in a direction determined by the sign of \(D\), which is the determinant of the matrix inverted; the rotation is counterclockwise if \(D > 0\), and it is clockwise if \(D < 0\) (according to Lemma 6).

With the above observations, and the fact that

\[
\begin{pmatrix}
p_1(s) \\
p_2(s)
\end{pmatrix} = \begin{pmatrix}
p_{10} \\
p_{20}
\end{pmatrix} + \frac{u^2}{2} \begin{bmatrix}
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2} \\
\frac{\partial^2 p_1}{\partial u^2} & \frac{\partial^2 p_2}{\partial u^2}
\end{bmatrix} \begin{bmatrix}
s = s_0 \\
p_1 = p_{10} \\
p_2 = p_{20}
\end{bmatrix} + o(u^2)
\]

we may describe the local mapping \((p_1(s), p_2(s))\) in a very informative manner when \(s - s_0\) is restricted to one quadrant. The situation for \(s - s_0\) in the first quadrant

\[
Q_1 = \{ s = s_0 + u e^{i\theta} \mid 0 < u < \delta, 0 \leq \theta \leq \pi/2 \}
\]

with \(D > 0\) is illustrated in Figure 2: the line segment \(C'E'\) (from \(s_0 + \delta i\) to \(s_0\)) is mapped to the curve \(C'E'\) in the \(p_{10}, p_{20}\) space, the arc \(EPB\) \((s = s_0 + \delta e^{i\theta}, 0 \leq \theta \leq \pi/2)\) is mapped to the curve \(E'P'B'\), and the line segment \(BC\) (from \(s_0 + \delta i\) to \(s_0\)) is mapped to the curve \(B'C'\).

In view of the second order derivatives, \(B'C'\) and \(C'E'\) have the same tangent at \(C'\). Continuity and local bijectivity (Corollary 4) imply that the singly connected region bounded by the line segments \(BC, C'E\) and the arc \(EPB\) is mapped by \((p_1(s), p_2(s))\) bijectively to the singly connected region bounded by the curves \(B'C', C'E'\) and \(E'P'B'\).

When \(D < 0\), the curve \(E'P'B'\) is roughly clockwise (instead of counterclockwise as in Figure 2) relative to the point \(C'\). The mapping with \(s - s_0\) in the other three quadrants are similar.

The complete mapping \((p_1(s), p_2(s))\) with \(s - s_0\) in all four quadrants may be divided into four possible cases depending on the sign of \(D\) and whether \(T_{(\omega_0, p_{10}, p_{20})}^-\) is on the counterclockwise or on the clockwise side of \(T_{(\omega_0, p_{10}, p_{20})}^+\) in the S-sector. The migration of the double roots in all cases is summarized in the following theorem.

Theorem 7 (Migration of Double Roots) If \((p_1, p_2)\) is in the G-sector in a sufficient small neighborhood of \((p_{10}, p_{20})\), then one root of (1) in the neighborhood of \(s_0\) is in the right half-plane, the other is in the left half-plane.

When \((p_1, p_2)\) is in the S-sector, then the two roots are either both in the left half-plane or both in the right half-plane. More specifically,

Case i. If \(D > 0\), and \(T_{(\omega_0, p_{10}, p_{20})}^-\) is on the counterclockwise side of \(T_{(\omega_0, p_{10}, p_{20})}^+\) in the S-sector, then both roots are on the left half-plane.

Case ii. If \(D > 0\), and \(T_{(\omega_0, p_{10}, p_{20})}^-\) is on the clockwise side of \(T_{(\omega_0, p_{10}, p_{20})}^+\) in the S-sector, then both roots are on
the right half-plane.

**Case iii.** If $D < 0$, and $\mathcal{T}_{(\omega_0,p_10,p_{20})}^-$ is on the counterclockwise side of $\mathcal{T}_{(\omega_0,p_10,p_{20})}^+$ in the S-sector, then both roots are on the right half-plane.

**Case iv.** If $D < 0$, and $\mathcal{T}_{(\omega_0,p_10,p_{20})}^-$ is on the clockwise side of $\mathcal{T}_{(\omega_0,p_10,p_{20})}^+$ in the S-sector, then both roots are on the right half-plane.

**Proof.** Consider Case i. The situation is illustrated in Figure 3. Let the region bounded by the arc $EBP$ and line segments $BC$ and $CE$ be denoted as $I$, and the region bounded by the curves $E'PB'$, $B'C'$ and $C'E'$ be denoted as $I'$. Similarly, region $II$ is bounded by $BQF$, $FC$, and $CB$, and region $II'$ is bounded by $B'QF'$, $F'C'$, $C'B'$; region $III$ is bounded by $FRA$, $AC$, and $CF$, and region $III'$ is bounded by $F'R'A'$, $A'C'$, $C'F'$; region $IV$ is bounded by $ASE$, $EC$, and $CA$, and region $IV'$ is bounded by $A'S'E'$, $E'C'$, $C'A'$. As discussed before the theorem, $(p_1(s), p_2(s))$ is a bijection from $I$ to $I'$ when $s$ is restricted to $I$. Similarly, $(p_1(s), p_2(s))$ is a bijection from $II$ to $II'$ when restricted to $II$, or from $III$ to $III'$ when restricted to $III$, or from $IV$ to $IV'$ when restricted to $IV$. As the S-sector (in a sufficiently small neighborhood) is contained in $I' \cap III'$, we may conclude that for any $(p_1, p_2)$ in the S-sector, one of the two characteristic roots in the neighborhood of $s_0$ must be in region $I'$, the other must be in region $III$, and obviously both in the left half-plane. Similarly, the G-sector (in a sufficiently small neighborhood) is contained in $(I' \cup IV') \cap (II' \cup IV')$. Therefore, for any $(p_1, p_2)$ in the G-sector, one of the two characteristic roots in the neighborhood of $s_0$ must be in $I' \cup IV'$ (in the right half-plane), and the other must be in $III \cup III'$ (in the left half-plane).

Case ii is illustrated in Figures 4. In this case, the S-sector is contained in $I' \cap IV'$, and therefore, the two characteristic roots in the neighborhood of $s_0$ must be in regions $I$ and $IV$, both in the right half-plane. The G-sector can still be expressed as $(I' \cup IV') \cap (II' \cup IV')$.

Case iii is illustrated in Figure 5, and Case iv is illustrated in Figure 6, and the conclusions can be drawn in a similar manner.

**5 Algebraic S-sector condition and global perspectives**

Theorem 7 indicates that the migration pattern of the two roots in the G-sector is always the same under the least degeneracy assumptions, which is the only case discussed in this article. However, judging the migration pattern of the two roots in the S-sector requires knowing the sign of $D$ and which side of $\mathcal{T}_{(\omega_0,T_{10},T_{20})}$ the curve...
\( T_{(\omega_0, p_{10}, p_{20})} \) is in the S-sector. Fortunately, by considering the third order derivatives, an explicit algebraic condition is possible.

**Corollary 8 (S-sector Criterion)** If \((p_1, p_2)\) is in the S-sector in a sufficiently small neighborhood of \((p_{10}, p_{20})\), then the two characteristic roots in the neighborhood of \(s_0\) are both in the left half-plane if

\[
\kappa < 0, \quad (24)
\]

where

\[
\kappa = \operatorname{Re} \left[ \frac{\partial^2 q}{\partial s^2} \left( - \frac{\partial^3 q}{\partial s^3} + 3 \frac{\partial^2 q}{\partial p_1 \partial s} \frac{\partial^2 p_1}{\partial s^2} + 3 \frac{\partial^2 q}{\partial p_2 \partial s} \frac{\partial^2 p_2}{\partial s^2} \right) \right]_{s=s_0, p_1=p_{10}, p_2=p_{20}},
\]

and \(\frac{\partial^2 p_i}{\partial s^2}\) may be evaluated by (23) or (22) with \(\gamma = i\). If

\[
\kappa > 0 \quad (25)
\]

instead, then both roots are in the right half-plane.

The proof is given in appendix A.

If \(\kappa = 0\), higher order derivatives may be used to evaluate conditions in Theorem 7.

It should be noticed that the roots of the characteristic equation discussed in Theorem 7 and Corollary 8 are restricted to a sufficiently small neighborhood of \(s_0 = j\omega_0\). Because characteristic roots are distributed symmetrically with respect to the real axis, there is also a double root at \(s_0^* = -j\omega_0\) when \(p_1 = p_{10}, p_2 = p_{20}\). When \((p_1, p_2)\) deviates from \((p_{10}, p_{20})\), the migration of the two roots in the neighborhood of \(s_0^*\) follows the same pattern as those in the neighborhood of \(s_0\).

There may also be roots on the imaginary axis outside of the neighborhoods of \(s_0\) and \(s_0^*\). The migration of these imaginary roots need to be analyzed separately.

Finally, the roots on the right half-plane remain on the right half-plane as long as \((p_1, p_2)\) stay within a sufficiently small neighborhood of \((p_{10}, p_{20})\). Similarly, the roots on the left half-plane remain on the left half-plane when the deviation of \((p_1, p_2)\) is sufficiently small.

### 6 Illustrative examples

In this section, we will present five examples of various nature to illustrate the application of the theory as well as some degenerate cases.

#### 6.1 Parameter-dependent polynomial

Polynomial characteristic equations are the most commonly seen in the undergraduate textbooks. An algebraic study on the stability of parameter-dependent polynomial can be found in Mailybaev (2000). A representative study on robust stability of parameter-dependent polynomials can be found in Barmish (1993). A Puiseux series approach has been used to study the perturbation of the multiple roots under small parameter deviation in Chen et al. (2010a) and Chen et al. (2010b). In the following example, we will use the method we have arrived to analyze such a system.

**Example 9** Consider the characteristic equation

\[
s^5 + p_1 s^4 + p_2 s^3 + p_1^2 s^2 + s + 2 = 0, \quad (26)
\]

where \(p_1\) and \(p_2\) are real parameters. For \((p_1, p_2) = (2, 2)\), systems (26) has double imaginary roots at \(s = \pm s_0 = \pm j\omega_0\), where \(\omega_0 = 1\). In addition, it has a root at \(-2\), which is in the left half-plane. The local stability crossing curve \(T_{(1,2,2)}\) is plotted in Figure 7, where \(C' A'\) is \(T_{(1,2,2)}\), and \(C' B'\) is \(T_{(1,2,2)}^+\). We can compute

\[
\kappa = -128 < 0.
\]

According to Corollary 8, this means that both roots at \(i\) moves to the left half-plane as \((p_1, p_2)\) moves into S-sector. Furthermore, we may compute

\[
D = 3 > 0.
\]

Therefore, the system (26) belongs to Case i of Theorem 7, i.e. \(T_{(\omega_0, p_{10}, p_{20})}\) is on the counterclockwise side of \(T_{(\omega_0, p_{10}, p_{20})}^+\) in the S-sector, which is consistent with Figure 7.

Also according to Theorem 7, as \((p_1, p_2)\) moves from \((2, 2)\) to the G-sector, one of the two imaginary roots at \(i\) moves to the right half-plane, and the other one moves to the left half-plane. The movement of the double roots at \(-i\) is symmetric to those at \(i\).

To summarize, for \((p_1, p_2) = (2, 2)\), the system has four roots on the imaginary axis and one root on the left half-plane. When \((p_1, p_2)\) moves into the S-sector, all five roots are on the left half-plane. When \((p_1, p_2)\) moves into the G-sector, there are two roots on the right half-plane, and the remaining three roots are on the left half-plane.

#### 6.2 Time-delay systems

Time-delay systems are widely used to model systems of various disciplines. Numerous classic examples of such systems can be found, for instance, in Kolmanovskii &
Myshkis (1999). More recent applications abound, see, for example, Atay (2013) for network systems. An accurate estimate of delays are often rather difficult, which made it especially important to analyze how the system stability may change as the delays deviate from the nominal values. One such study for systems with two delays can be found in Gu et al. (2005). In the following example, we will apply the results in this paper on such a system.

**Example 10** Consider a time-delay system with the following characteristic quasi-polynomial

\[ q(s) = s^2 - 2s + 2 + [(2 \cos 1) s - 2 (\cos 1 + \sin 1)] e^{-\tau_1 s} + e^{-\tau_2 s} \tag{27} \]

Note that the function given in (27) is of the form (2), with polynomials \( r_0 \), \( \tau_1 \) and \( \tau_2 \) of order 2, 1, and 0, respectively. For \((\tau_1, \tau_2) = (1, 2)\), system (27) has double imaginary roots at \( s = s_0 = \pm i\omega_0 \) with \( \omega_0 = 1 \). It can be computed that

\[ D \approx 1.74159 > 0 \]

\[ \kappa \approx 30.7082 > 0 \]

From the sign of \( D \) and \( \kappa \), it can be concluded that this system belongs to Case ii of Theorem 7, i.e., \( T_{(\omega_0, \tau_10, \tau_20)} \) is on the clockwise side of \( T_{(\omega_0, \tau_10, \tau_20)}^+ \) in the S-sector.

The stability crossing curve \( T \) (which contains both positive and negative local stability crossing curves) is plotted in Figure 8. It can be seen that \( T \) divides the region into three regions: region A is connected to the origin, region B is the small region on the upper side, and region C is the small region on the lower side. For \( \tau_1 = 0, \tau_2 = 0 \), the characteristic quasipolynomial is reduced to a polynomial, and it can be easily calculated that both roots are on the right half-plane. Therefore, the region connected to the origin has two right half-plane roots.

According to Corollary 8 or Theorem 7, both imaginary roots move to the right half-plane as \((\tau_1, \tau_2)\) moves to the S-sector (which is connected to region B). According to Theorem 7, as \((\tau_1, \tau_2)\) moves to the G-sector (which is connected to region A), one of the two imaginary roots moves to the right half-plane, the other moves to the left half-plane. In other words, as \((\tau_1, \tau_2)\) moves from region B to region A through (1, 2), one root moves from the right half-plane to the left half-plane passing through the point \( i \) on the imaginary axis, another root on the right half-plane moves to touch the imaginary axis at \( i \) then return to the right half-plane.

Due to symmetry, another left half-plane root moves to the right half-plane through the point \(-i\). Therefore, there are two more right half-plane roots when \((\tau_1, \tau_2)\) is in region B as compared to the case when \((\tau_1, \tau_2)\) is in region A. Thus, we conclude that there are four roots on the right half-plane when \((\tau_1, \tau_2)\) is in region B.

For region C, it can be easily calculated using the method described in Gu et al. (2005) that the two right half-plane roots cross the imaginary axis to the left half-plane as \((\tau_1, \tau_2)\) moves from region A. Therefore, there is no right half-plane root in this region, and the system is stable for \((\tau_1, \tau_2)\) in region C.
model population dynamics as follows. An early example is given by Cushing (1977) to illustrate degenerate cases. The first example shows the local stability crossing curve may not have a cusp when one of the least degeneracy assumptions, \( D \neq 0 \), is violated. The second example shows that the S-sector may be empty.

### Example 12
Consider the characteristic equation

\[
 s^5 + s^4 + p_2 s^3 + (p_1 + 1) s^2 + s + p_1 = 0,
\]

where \( p_1 \) and \( p_2 \) are real parameters. For \( p_1 = 1 \) and \( p_2 = 2 \), (30) has a double root at \( s_0 = \pm i \omega_0 \) with \( \omega_0 = 1 \).
that depend on two parameters is studied under the least

7 Concluding remarks

We can compute \( D = 0 \), and therefore (14) is violated.

The local stability crossing curve is plotted in Figure 10. It can be seen that there is no cusp at (1, 2), and S-sector and G-sector are not well defined.

Example 13 Consider the characteristic equation

\[
s^4 + (p_1 + p_2) s^3 + 2 (p_1 p_2 + 2) s^2 + (p_1 + p_2) s + p_1 p_2 + \frac{7}{4} = 0, \quad (31)
\]

with two parameters, \( p_1 \) and \( p_2 \).

For \((p_1, p_2) = (p_{10}, p_{20}) = (\sqrt{\frac{3}{2}}, -\sqrt{\frac{3}{2}})\), system (31) has double imaginary characteristic roots at \( s = \pm s_0 = \pm i \omega_0 \) for \( \omega_0 = \frac{\sqrt{3}}{2} \). We compute \( D \) and \( \kappa \) and obtain

\[
D = \frac{\sqrt{3}}{4} > 0
\]

\[
\kappa = -96\sqrt{2} < 0
\]

We note that in a neighbourhood of \((p_{10}, p_{20})\) we can easily confirm that \( p_1 = p_2 \) for \( p_1 \leq p_{10} \) results in two imaginary roots in the neighbourhood of \( s_0 \). This means that the positive and negative local stability crossing curves, \( T_{(\omega_0, p_{10}, p_{20})} \) and \( T_{(\omega_0, p_{10}, p_{20})} \), coincides and the S-sector is empty. This situation is depicted in Figure 11. However, the conclusion about the G-sector still holds, i.e. there is a characteristic root in the right half-plane in the neighbourhood of \( -s_0 \), and another one in the neighbourhood of \( -s_0 \), as \((p_1, p_2)\) moves to the G-sector.

A Proof of Corollary 8

Differentiate (21) with respect to \( u \), we obtain

\[
\begin{align*}
\frac{\partial^3 q}{\partial p_1^3} \left( \frac{\partial p_1}{\partial u} \right)^3 & + 3 \frac{\partial^3 q}{\partial p_1^3} \left( \frac{\partial p_1}{\partial u} \right)^2 \frac{\partial p_2}{\partial u} + \\
+3 \frac{\partial^3 q}{\partial p_1^3} \left( \frac{\partial p_1}{\partial u} \right) & ^2 + 3 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial p_1}{\partial u} \frac{\partial^2 p_1}{\partial u^2} + \\
+3 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} & ^2 + 3 \frac{\partial q}{\partial u} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} \\
+6 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} & \gamma + 3 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial^2 p_1}{\partial u^2} + \\
+3 \frac{\partial q}{\partial u} \frac{\partial p_1}{\partial u} \frac{\partial p_2}{\partial u} & \gamma^2 + 3 \frac{\partial q}{\partial u} \frac{\partial^2 p_1}{\partial u^2} + \\
+3 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial^2 p_1}{\partial u^2} & \gamma + 3 \frac{\partial^2 q}{\partial p_1^2} \frac{\partial^2 p_1}{\partial u^2} + \\
+3 \frac{\partial^2 q}{\partial p_2^2} \gamma + & 3 \frac{\partial^2 q}{\partial p_2^2} \gamma^2 + 3 \frac{\partial^2 q}{\partial p_2^2} \gamma^3 = 0.
\end{align*}
\] (A.1)
Let $s = s_0$ \( \frac{\partial p_1}{\partial p_1}^{p_1=p_{10}} \frac{\partial p_2}{\partial p_2}^{p_2=p_{20}} \), $k = 1, 2, 3$, and

$$
\frac{\partial^k p_1}{\partial x^k} = \frac{\partial^k p_1}{\partial y^k}^{p_1=p_{10}}, \frac{\partial^k p_2}{\partial y^k}^{p_2=p_{20}}, \text{ for } k = 1, 2, 3.
$$

Then Taylor series gives

$$
\begin{align*}
\left( \frac{\partial p_1}{\partial u} \right)_0 & = \left( \frac{\partial p_1}{\partial u} \right)_0 + \delta \frac{\partial^2 p_1}{\partial u^2} \delta + \frac{\delta^2}{2} \frac{\partial^3 p_1}{\partial u^3} \delta^2 + \frac{\delta^3}{6} \frac{\partial^4 p_1}{\partial u^4} \delta^3 + o(\delta^4). \\
\left( \frac{\partial p_2}{\partial u} \right)_0 & = \left( \frac{\partial p_2}{\partial u} \right)_0 + \frac{\delta}{2} \frac{\partial^2 p_2}{\partial u^2} \delta + \frac{\delta^2}{6} \frac{\partial^3 p_2}{\partial u^3} \delta^2 + \frac{\delta^3}{6} \frac{\partial^4 p_2}{\partial u^4} \delta^3 + o(\delta^4).
\end{align*}
$$

But according to (19) and (22), we have

$$
\begin{align*}
\left( \frac{\partial p_1}{\partial u} \right)_0 & = 0, \\
\left( \frac{\partial^2 p_1}{\partial u^2} \right)_0 & = \left( \frac{\partial^2 p_2}{\partial u^2} \right)_0 = 0.
\end{align*}
$$

Therefore,

$$
\begin{align*}
\left( \frac{\partial p_1}{\partial u} \right)_0 & = \left( \frac{\partial p_2}{\partial u} \right)_0 = 0, \\
\left( \frac{\partial^2 p_1}{\partial u^2} \right)_0 & = \left( \frac{\partial^2 p_2}{\partial u^2} \right)_0 = 0.
\end{align*}
$$

As the tangent direction of the local stability crossing curve $\mathcal{T}_{(\omega_{0,0},p_{10},p_{20})}$ at the cusp ($p_{10}, p_{20}$) is $\left( \frac{\partial p_1}{\partial u}, \frac{\partial p_2}{\partial u} \right)^T$, it can be easily seen that the $\mathcal{T}_{(\omega_{0,0},p_{10},p_{20})}$ is in the counterclockwise side of $\mathcal{T}_{(\omega_{0,0},p_{10},p_{20})}$ if we may reach the direction of $\left( \frac{\partial p_1}{\partial u}, \frac{\partial p_2}{\partial u} \right)^T$ by rotating ($\Delta p_1, \Delta p_2$) counterclockwise through an angle $\theta$ \( (0, \pi) \) as is shown in Figure A.1. Let $\left( -\frac{\partial p_1}{\partial u} \right)_0 = \left( -\frac{\partial p_2}{\partial u} \right)_0 = 0$. Comparing the expressions (A.4) and (22) and using Lemma 6, we
can see that the above can be achieved if we can reach the direction of \(-\frac{\partial^2 q}{\partial s^2}\) by rotating \(\Delta B\) counterclockwise through an angle of \(\theta \in (0, \pi)\) if \(D > 0\) (which is Case i in Theorem 7). The rotation from \(\Delta B\) to \(-\frac{\partial^2 q}{\partial s^2}\) needs to be clockwise if \(D < 0\) (which is Case iii). The counterclockwise rotation from \(\Delta B\) to \(-\frac{\partial^2 q}{\partial s^2}\) may be expressed as

\[
\text{Re}(\Delta B) \text{Im} \left( -\frac{\partial^2 q}{\partial s^2} \right) - \text{Im}(\Delta B) \text{Re} \left( -\frac{\partial^2 q}{\partial s^2} \right) > 0,
\]

which is equivalent to (24), and the conclusion is valid in this case in view of Case i in Theorem 7. It can be similarly shown that if we can rotate \(\Delta B\) to the direction of \(-\frac{\partial^2 q}{\partial s^2}\) clockwise through an angle of \(\theta \in (0, \pi)\), then (25) is satisfied, and the conclusion is valid in this case also in view of Case iii in Theorem 7.

Similarly, we can show that \(\kappa > 0\) and \(D > 0\), or \(\kappa < 0\) and \(D < 0\) can guarantee that we can reach the direction of \(-\frac{\partial^2 q}{\partial s^2}\), \(-\frac{\partial^2 q}{\partial s^2}\), clockwise through an angle \(\theta \in (0, \pi)\), and the conclusions are true in view of Case ii and Case iv in Theorem 7. We have exhausted all possibilities, and the corollary is proven.

References


