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Zeons, Orthozeons, and Graph Colorings

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Abstract

Nilpotent adjacency matrix methods have proven useful for counting self-avoiding walks (paths, trails, cycles, and circuits) in finite graphs. In the current work, these methods are extended for the first time to problems related to graph colorings. Nilpotent-algebraic formulations of graph coloring problems include necessary and sufficient conditions for k-colorability, enumeration (counting) of heterogeneous and homogeneous paths, trails, cycles, and circuits in colored graphs, and a matrix-based greedy coloring algorithm. Introduced here also are the *orthozeons* and their application to counting monochromatic self-avoiding walks in colored graphs. The algebraic formalism easily lends itself to symbolic computations, and *Mathematica*-computed examples are presented throughout.

Keywords: zeons, orthozeons, heterochromatic, monochromatic, paths, cycles, trails, circuits

1 Introduction

Nilpotent adjacency matrix methods were first developed by Staples for counting self-avoiding walks (paths, cycles, trails, & circuits) in finite graphs [28]. Weighting the vertices of a graph with zeon generators allows one to construct a *nilpotent adjacency matrix*, $\mathfrak{A}$, whose entries are generators of the algebra. The matrix is very convenient for performing symbolic computations and allows enumeration of cycles by considering traces of matrix powers. This idea has led to a number of applications to graph enumeration problems and even routing problems in communication networks [3, 8, 16, 24].

Zeon algebras can be thought of as commutative analogues of fermion algebras, which are isomorphic to Clifford algebras of appropriate signature. In the Clifford algebra context, properties of spinors have been applied to the study of maximal cliques [5] and an assortment of other graph-theoretic problems [14]. Although Clifford algebras provide the underlying framework for everything appearing here, the broader Clifford algebra formalism is unnecessary for matters at hand.

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Combinatorial properties of zeons have been shown to generate Stirling numbers of the second kind, Bell numbers, Catalan numbers, and Bessel numbers [22]. Further, they have been useful in defining partition-dependent stochastic integrals. In fact, expanding powers of zeon elements is equivalent to summing over partitions [23].

Recently, combinatorial identities involving zeons have been studied in a number of works by A.F. Neto [17, 18, 19, 20]. In these works, Bernoulli numbers, m-Stirling numbers of the second kind, higher order derivatives of trigonometric functions, and representations of Bernoulli polynomials are presented in the context of zeon algebras.

This paper is the first extension of nilpotent adjacency matrix methods to graph colorings. Algebraic formulations of k-colorability are presented for both vertex- and edge-colored graphs. Powers of nilpotent coloring matrices are used to count *heterochromatic* paths and cycles in vertex-colored graphs (trails and circuits in edge-colored graphs). Further, by considering the semigroup of orthogonal rank-one projections, the approach is extended to count *monochromatic* self-avoiding walks in colored graphs.

Monochromatic paths and cycles have been objects of interest for decades, and a detailed survey of results would be impractical. Notable works include the papers of Erdös and Tuza [10, 11], Albert, Frieze and Reed [1, 12], and Broersma, *et al.* [4].

In 1973, Raynaud proved a conjecture by Lehel that a 2-colored complete symmetric directed graph with at least two vertices contains a simple directed (monochromatic) Hamiltonian cycle. In a 1983 paper, Gyáfás surveyed results covering the vertices of 2-colored complete graphs by two paths or two cycles of different color [13]. In [15], Li, Zhang, and Broersma established some sufficient conditions for the existence of monochromatic and heterochromatic paths and cycles.

In [6], Chen and Li assume that the *color degree* [1] of a graph's vertices is bounded below by some integer k, and show that if $3 \leq k \leq 7$, then G has a heterochromatic path of length at least $k-1$. They also show that if $k > 8$, then G has a heterochromatic path of length at least $\lceil \frac{3k}{5} \rceil + 1$.

More recently, Babu, Chandran, and Rajendraprasad [2] established lower bounds for the length of a maximum heterochromatic path in an edge colored graph without small cycles.

Beyond the utility of nilpotent adjacency matrix methods in graph enumeration problems [28] and routing problems in networks and multi-constrained path problems [3, 8, 16], the extension of these methods to graph colorings allows one to count heterochromatic and monochromatic self-avoiding walks in colored graphs. Moreover, the zeon-algebraic formalism allows one to quickly verify whether a given graph coloring is proper, and provides a convenient framework for implementing greedy coloring algorithms.

The rest of the paper is laid out as follows. Essential definitions and terminol-

[1] The color degree of a vertex is the cardinality of the distinct colors among the vertex's neighbors.

ogy related to the zeon algebra, denoted ${\mathcal{C}\ell_n}^{\rm nil}$ appear in Subsection 1.1. After reviewing essential terminology and notation from graph theory in Subsection 1.2, results involving verification of proper colorings and counting *heterochromatic* self-avoiding walks are discussed in Section 2. A matrix formulation of a greedy vertex-coloring algorithm is presented in Section 3.

Orthozeons are introduced in Section 4, where they are defined and used to construct orthozeon vertex- and edge-coloring matrices associated with finite graphs. These matrices are then applied to the counting of *monochromatic* self-avoiding walks in finite graphs.

Graph polynomials having zeon and orthozeon coefficients are defined in Section 5. Properties of these polynomials reveal information about the heterochromatic (or monochromatic) circumference of a colored graph and the size of a maximal heterochromatic (or monochromatic) matching.

Illustrative *Mathematica* examples are presented throughout the paper. The interested reader can find examples and necessary code through the *Research* link at `http://www.siue.edu/~sstaple`. The paper concludes in Section 6 with a discussion of avenues for further research.

1.1 Zeon Preliminaries

Let ${\mathcal{C}\ell_n}^{\rm nil}$ denote the real abelian algebra generated by the collection $\{\zeta_i : 1 \leq i \leq n\}$ along with the scalar $1 = \zeta_0$ subject to the following multiplication rules:

$$\zeta_i\,\zeta_j = \zeta_j\,\zeta_i \quad \text{for } i \neq j, \text{ and}$$
$${\zeta_i}^2 = 0 \quad \text{for } 1 \leq i \leq n.$$

It is evident that a general element $\alpha \in {\mathcal{C}\ell_n}^{\rm nil}$ has canonical expansion of the form $\alpha = \sum_{I \in 2^{[n]}} \alpha_I\,\zeta_I$. Here, $I \in 2^{[n]}$ is a subset of $[n] = \{1, 2, \ldots, n\}$, used as a multi-index, $\alpha_I \in \mathbb{R}$, and $\zeta_I = \prod_{\iota \in I} \zeta_\iota$. The algebra ${\mathcal{C}\ell_n}^{\rm nil}$ is called the (n-particle) *zeon algebra*, and the generators $\{\zeta_i : 1 \leq i \leq n\}$ are referred to simply as *zeons*.

As a vector space, this 2^n-dimensional algebra has a canonical basis of *basis blades* of the form $\{\zeta_I : I \subseteq [n]\}$. The null-square property of the generators $\{\zeta_i : 1 \leq i \leq n\}$ guarantees that the product of two basis blades satisfies the following:

$$\zeta_I\zeta_J = \begin{cases} \zeta_{I\cup J} & I \cap J = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

An inner product is defined on ${\mathcal{C}\ell_n}^{\rm nil}$ by linear extension of

$$\left\langle \sum_{I \in 2^{[n]}} u_I\zeta_I, \zeta_J \right\rangle = u_J.$$

Hence, $u \in \mathcal{C}\ell_n{}^{\text{nil}}$ implies $u = \sum_{I \in 2^{[n]}} \langle u, \zeta_I \rangle \zeta_I$. Finally, the sum of the scalar coefficients in the canonical expansion of u is called the *scalar sum* of u and is denoted by $\langle\langle u \rangle\rangle$. In particular, when $u = \sum_{I \in 2^{[n]}} u_I \zeta_I$, the scalar sum is given by $\langle\langle u \rangle\rangle = \sum_{I \in 2^{[n]}} u_I$.

1.2 Graph Theory Background

The terminology appearing here is more-or-less standard, and can be found in any number of graph theory texts. The reader is referred to [29] for graph theory beyond the essential notation and terminology found here.

A *graph* $G = (V, E)$ is a collection of vertices V and a set E of unordered pairs of vertices called *edges*. A *directed graph* is a graph whose edges are *ordered pairs* of vertices.

Two vertices $v_i, v_j \in V$ are said to be *adjacent* if there exists an edge $e = (v_i, v_j) \in E$. In this case, the edge e is said to be *incident* to vertices v_i and v_j. The number of edges incident to a vertex is called the *degree* of the vertex. Two edges are said to be *coincident* if they are incident to a common vertex.

A graph is *finite* if V and E are finite sets, that is, if $|V|$ and $|E|$ are finite numbers. A *loop* in a graph is an edge of the form (v, v). A graph is said to be *simple* if it is undirected, contains no loops, and no unordered pair of vertices appears more than once in E.

A *k-walk* $\{v_0, \ldots, v_k\}$ in a graph G is a sequence of vertices in G with *initial vertex* v_0 and *terminal vertex* v_k such that there exists an edge $(v_j, v_{j+1}) \in E$ for each $0 \leq j \leq k-1$. A k-walk contains k edges. A *k-path* is a k-walk in which no vertex appears more than once. A *k-trail* is a k-walk in which no edge appears more than once. A *closed k-walk* is a k-walk whose initial vertex is also its terminal vertex. A *k-cycle* is a closed k-path (with the exception $v_0 = v_k$), while a *k-circuit* is a closed k-trail. For purposes of the current work, *self-avoiding walks* include trails, paths, circuits and cycles, as determined by context.

Given a graph $G = (V, E)$ on n vertices, the *adjacency matrix* $A = (a_{ij})$ of G is defined by

$$a_{ij} = \begin{cases} 1 & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 1.1. A (*vertex*) *coloring* of a graph $G = (V, E)$ is a mapping $\phi : V \to \{1, \ldots, \kappa\}$. The set $\{1, \ldots, \kappa\}$ is referred to as the *palette* of the coloring, and its elements are referred to as *colors*. A coloring ϕ is said to be *proper* if $(v_i, v_j) \in E$ implies $\phi(v_i) \neq \phi(v_j)$.

In light of Definition 1.1, a *colored graph* is a pair (G, ϕ), where ϕ is a coloring of the graph G. For purposes of the current paper, a *surjective* mapping $\phi : V \to \{1, \ldots, \kappa\}$ will be referred to as a vertex κ-coloring.

Definition 1.1 extends naturally to *edge colorings*. A *proper edge coloring* is a mapping $\theta : E \to \{1, \ldots, \kappa\}$ such that $(v_i, v_\ell), (v_\ell, v_j) \in E$ implies $\theta((v_i, v_\ell)) \neq \theta((v_\ell, v_j))$. In other words, no pair of coincident edges can be associated with the same color in a proper edge coloring.

A κ-coloring that is proper will be referred to specifically as a *proper κ-coloring*. A graph G will be said to be *properly κ-colorable* if there exists a proper coloring of G having a palette of cardinality κ. The minimal κ for which a proper κ-coloring exists is called the *chromatic number* of G.

2 Proper Colorings and Heterochromatic Walks

The motivation behind the application of zeon algebras to graph colorings is the nilpotent adjacency matrix approach to counting self-avoiding walks, as developed in [28]. To briefly review, let $G = (V, E)$ be a graph on n vertices (either simple or directed with no multiple edges), and let $\{\zeta_i : 1 \leq i \leq n\}$ denote the nilpotent generators of ${\mathcal{C}\ell_n}^{\rm nil}$. The *nilpotent adjacency matrix* associated with G is defined by

$$\langle v_i|\mathcal{A}|v_j\rangle = \begin{cases} \zeta_j \text{ if } (v_i, v_j) \in E \\ 0 \text{ otherwise.} \end{cases}$$

When $\mathcal{A}$ is the nilpotent adjacency matrix of an n-vertex graph G, it is not difficult to show (by induction) that for positive integer m,

$$\langle\langle \mathrm{tr}(\mathcal{A}^m)\rangle\rangle = m\, X_m,$$

where X_m denotes the number of m-cycles appearing in the graph G. By considering off-diagonal entries of $\mathcal{A}^m$ one can also determine the number of m-paths between a given initial-terminal pair of vertices. Furthermore, the approach extends naturally to counting trails and circuits by a simple modification to the construction of $\mathcal{A}$.

With this motivation in mind, the task now is to define a number of nilpotent matrices associated with vertex- and edge-colorings of a finite graph. Properties of these matrices will be used to quickly determine whether or not a given graph coloring is proper and to count the heterochromatic self-avoiding walks (i.e., paths, cycles, trails, & circuits) in a finite graph.

Definition 2.1. Let $G = (V, E)$ be a graph on n vertices with vertex κ-coloring ϕ. The *zeon vertex-coloring matrix* Ψ *associated with* (G, ϕ) is the $n \times n$ matrix having entries in ${\mathcal{C}\ell_\kappa}^{\rm nil}$ defined by

$$\langle v_i|\Psi|v_j\rangle = \begin{cases} \zeta_{\phi(v_j)} \text{ if} (v_i, v_j) \in E \\ 0 \text{ otherwise.} \end{cases}$$

Theorem 2.2. *Let Ψ be the zeon vertex-coloring matrix of a colored graph (G, ϕ) on n vertices. Then, for $1 \leq i, j \leq n$ and $k \in \mathbb{N}$,*

$$\langle v_i | \Psi^k | v_j \rangle = \sum_{|I|=k} \alpha_I \zeta_I$$

where α_I is the number of walks from $v_i \to v_j$ in the graph such that the vertex colors are indexed by I and no color is repeated with the possible exception of $\phi(v_i)$; in particular, no vertex can be repeated, except the initial vertex can repeated at most once in an intermediate step.

Proof. Proof is by using induction on $k \geq 1$. When $k = 1$, the result holds by definition of Ψ. Let a *good walk* $v_i \to v_\ell$ on I be a walk from v_i to v_ℓ such that no vertex color is repeated except possibly $\phi(v_i)$ one time. Now, assume the result holds for some $k \geq 1$.

$$\begin{aligned}
\langle v_i | \Psi^{k+1} | v_j \rangle &= \langle v_i | \Psi^k \Psi | v_j \rangle \\
&= \sum_{l=1}^{n} \langle v_i | \Psi^k | v_\ell \rangle \langle v_\ell | \Psi | v_j \rangle \\
&= \sum_{l=1}^{n} \sum_{|I|=k} \sharp\{k \text{ good walks } v_i \to v_\ell \text{ on I}\} \zeta_I \langle v_\ell | \Psi | v_j \rangle.
\end{aligned}$$

For fixed ℓ, $\langle v_i | \Psi^k | v_\ell \rangle$ is a linear combination of ζ_I's representing good k-walks from v_i to v_ℓ, the product of an arbitrary term from this linear combination with $\langle v_\ell | \Psi | v_j \rangle$ will be zero if there is no edge from v_ℓ to v_j or if v_j is the same color as another vertex in the good k-walk $v_i \to v_\ell$. Hence, the product $\langle v_i | \Psi^k | v_\ell \rangle \langle v_\ell | \Psi | v_j \rangle$ represents a sum of good $(k+1)$-walks $v_i \to v_j$ of the following form:

$$\sum_{|I|=k} \sharp\{\text{good } k\text{-walks } v_i \to v_\ell \text{ on I}\} \sharp\{1\text{-walks } v_\ell \to v_j\} \zeta_I \zeta_{\phi(v_j)} =$$

$$\sum_{|I|=k+1} \sharp\{\text{good } (k+1)\text{-walks } v_i \to v_j \text{ on I visiting } v_\ell \text{ in step } k\} \zeta_{I \cup \phi(v_j)}.$$

So, summing over all ℓ represents the colors along a path, as long as no colors are repeated and v_i can be visited only once after starting the path. □

Counting heterochromatic cycles is accomplished by the following corollary.

Corollary 2.3. *Let Ψ be the zeon vertex-coloring matrix of a colored graph (G, ϕ) on n vertices. Let $k \in \mathbb{N}$ be arbitrary and let $\mathfrak{h}_k$ denote the number of heterochromatic k-cycles in (G, ϕ). Then, $\langle\langle \mathrm{tr}(\Psi^k) \rangle\rangle = k\mathfrak{h}_k$.*

Proof. The result follows from Theorem 2.2 by making two observations. First, the last vertex visited in a cycle is the initial vertex, so any walk revisiting the initial vertex in an intermediate step will be annihilated by closing the walk. Secondly, each k-cycle appears along the main diagonal with multiplicity k due to various choices of basepoint for the cycle. □

Theorem 2.2 also reveals an algebraic method for determining whether a coloring is proper.

Corollary 2.4. *A zeon vertex-coloring matrix* Ψ *represents a proper coloring of a graph* $G = (V, E)$ *if and only if*

$$\left\langle\left\langle \mathrm{tr}(\Psi^2) \right\rangle\right\rangle = 2|E|.$$

Proof. By Theorem 2.2, $\langle v_i|\Psi^2|v_i\rangle$ is a linear combination of products of zeon pairs representing heterochromatic 2-cycles based at v_i. Each 2-cycle appears twice in the trace by choice of basepoint. It follows immediately that the scalar sum $\left\langle\left\langle \mathrm{tr}\,\Psi^2 \right\rangle\right\rangle$ is twice the number of heterochromatic pairs of adjacent vertices in the graph. By definition, the graph is properly colored if and only if every adjacent pair of vertices is heterochromatic. □

Corollary 2.4 can be restated to provide the following nilpotent adjacency matrix formulation of k-colorability of a graph.

Theorem 2.5 (Proper κ-colorability)**.** *A graph* $G = (V, E)$ *is* properly κ-colorable *if and only if there exists a nilpotent coloring matrix* Ψ *having entries in* ${\mathcal{C}\ell_\kappa}^{\mathrm{nil}}$ *such that*

$$\left\langle\left\langle \mathrm{tr}(\Psi^2) \right\rangle\right\rangle = 2|E|.$$

Example 2.6. Vertices of the graph seen in Figure 1 were colored with 8 randomly-assigned colors. The trace of ψ^2 as computed by *Mathematica* is

$$\begin{aligned} &6\zeta_{\{1,2\}} + 8\zeta_{\{1,3\}} + 2\zeta_{\{1,5\}} + 2\zeta_{\{1,6\}} + 2\zeta_{\{1,7\}} + 4\zeta_{\{1,8\}} + 2\zeta_{\{2,3\}}\\ &+4\zeta_{\{2,4\}} + 2\zeta_{\{2,5\}} + 4\zeta_{\{2,6\}} + 4\zeta_{\{2,7\}} + 2\zeta_{\{2,8\}} + 6\zeta_{\{3,4\}} + 4\zeta_{\{3,5\}}\\ &+2\zeta_{\{3,7\}} + 6\zeta_{\{3,8\}} + 2\zeta_{\{4,5\}} + 2\zeta_{\{4,6\}} + 2\zeta_{\{4,7\}} + 4\zeta_{\{4,8\}} + 2\zeta_{\{5,6\}}\\ &\qquad +4\zeta_{\{5,7\}} + 2\zeta_{\{5,8\}} + 4\zeta_{\{6,8\}} + 8\zeta_{\{7,8\}}. \end{aligned}$$

The scalar sum of the trace is 90, while the graph contains 54 edges. Hence, the coloring represented by ψ is not a proper coloring.

The previous theorems and corollaries can now be extended from vertex colorings to edge colorings.

Definition 2.7. Let $G = (V, E)$ be a graph (either simple or directed with no multiple edges) on n vertices with edge κ-coloring θ. Define the *zeon edge-coloring matrix* Λ associated with (G, θ) to be the $n \times n$ matrix having entries in ${\mathcal{C}\ell_\kappa}^{\mathrm{nil}}$ given by

$$\langle v_i|\Lambda|v_j\rangle = \begin{cases} \zeta_{\theta(v_i,v_j)} & (v_i, v_j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 2.8. *Let* Λ *be the zeon edge-coloring matrix of a simple graph of* G *on* n *vertices. Then, for* $1 \le i, j \le n$ *and* $m \in \mathbb{N}$,

$$\langle v_i|\Lambda^m|v_j\rangle = \sum_{|I|=m} \alpha_I \zeta_I$$

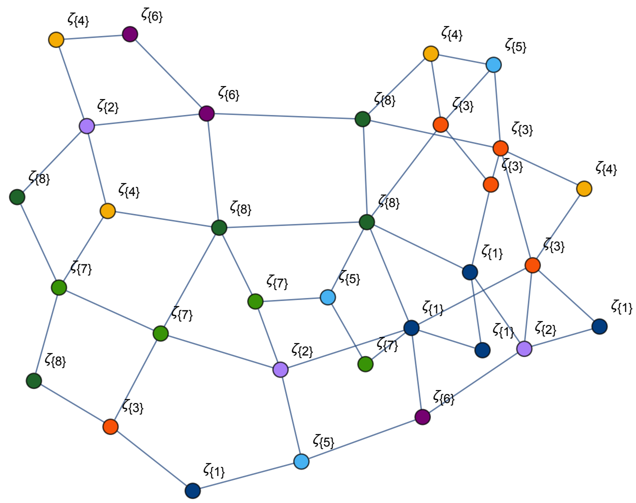

Figure 1: A vertex-colored graph on 32 vertices.

where α_I is the number of trails from $v_i \to v_j$ in the graph such that the edge colors are indexed by I and no color is repeated. In particular, α_I is the number of heterochromatic trails from $v_i \to v_j$ on colors indexed by I.

Proof. Proof is by using induction on $m \geq 1$. When $m = 1$, the result holds by definition of Λ. Assume the result holds for some $m \geq 1$.

$$\begin{aligned}
\langle v_i|\Lambda^{m+1}|v_j\rangle &= \langle v_i|\Lambda^m\Lambda|v_j\rangle \\
&= \sum_{l=1}^{n}\langle v_i|\Lambda^m|v_\ell\rangle\langle v_\ell|\Lambda|v_j\rangle \\
&= \sum_{l=1}^{n}\sum_{|I|=m}\sharp\{\text{good } m\text{-trails } v_i \to v_\ell \text{ on I}\}\zeta_I\langle v_\ell|\Lambda|v_j\rangle.
\end{aligned}$$

For fixed ℓ, $\langle v_i|\Lambda^m|v_\ell\rangle$ is a linear combination of ζ_I's representing good (i.e., heterochromatic) m-trails from v_i to v_ℓ. The product of any term from this linear combination with $\langle v_\ell|\Lambda|v_j\rangle$ will be zero if there is no edge from v_ℓ to v_j or if the edge (v_ℓ, v_j) is the same color as another edge in the m-trail $v_i \to v_\ell$. Hence, the product $\langle v_i|\Lambda^k|v_\ell\rangle\langle v_\ell|\Lambda|v_j\rangle$ represents a sum of $(m+1)$-trails $v_i \to v_j$ of the following form:

$$\sum_{|I|=m}\sharp\{m\text{-trails } v_i \to v_\ell \text{ on colors indexed by } I\}\zeta_I\zeta_{\theta(v_\ell,v_j)} =$$

$$\sum_{|I|=m+1}\sharp\{(m+1)\text{-trails } v_i \to v_j \text{ on colors } I\text{; last step } (v_\ell, v_j)\ \}\zeta_{I\cup\theta(v_\ell,v_j)}.$$

Summing over all ℓ gives a representation of all heterochromatic $(m+1)$-trails $v_i \to v_j$ in G. $\square$

Corollary 2.9. *Let Λ be a zeon edge-coloring matrix of a graph G. For any $m \in \mathbb{N}$,*

$$\langle\langle \mathrm{tr}(\Lambda^m) \rangle\rangle = m\rho,$$

where ρ is the number of heterochromatic m-circuits in G.

Proof. From Theorem 2.8, element $\langle v_i|\Lambda^m|v_i\rangle$ is a linear combination of heterochromatic m-circuits based at v_i. Each m-circuit appears with multiplicity m along the main diagonal due to the possible choices of basepoint. Hence, the result. $\square$

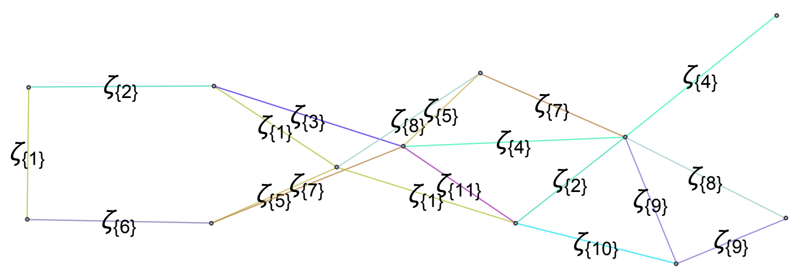

Figure 2: A zeon 11-edge-colored graph on 12 vertices.

In order to formulate a way to determine proper colorings, there needs to be a checking of all coincident pairs of edges to be sure no two coincident edges are the same color.

Lemma 2.10. *The total number of pairs of coincident edges in a graph G is given by*

$$\sum_{v \in V} \binom{\deg(v)}{2}.$$

Proof. First, note that if $\deg(v) \leq 1$ for some $v \in V$, there is no pair of edges coincident with v. By definition, $\binom{\deg(v)}{2} = 0$ in this case. Otherwise, $\binom{\deg(v)}{2}$ represents the number of pairs of edges coincident with v. Summing over all vertices thus gives the result. $\square$

Recalling that entries of Λ^2 represent heterochromatic 2-trails and 2-circuits in G, off-diagonal elements correspond to heterochromatic pairs of coincident edges. For convenience, set $\beta = (1, 1, \ldots, 1) \in \mathbb{R}^n$ for appropriate dimension n, determined henceforth by context.

Theorem 2.11. *The zeon edge-coloring matrix* Λ *represents a proper edge coloring of a graph* $G = (V, E)$ *if and only if*

$$\langle\langle \beta\Lambda^2\beta^\dagger \rangle\rangle = 2\sum_{v\in V}\binom{\deg(v)}{2}.$$

Proof. By Theorem 2.8, $\langle v_i|\Lambda^2|v_j\rangle$ is a linear combination of ζ_Is representing the sum of all heterochromatic 2-trails $v_i \to v_j$. It is clear that heterochromatic 2-circuits cannot exist in any edge-colored graph, so the diagonal entries of Λ^2 are all zero. Summing coefficients of all off-diagonal entries is accomplished by computing $\langle\langle \beta\Lambda^2\beta^\dagger \rangle\rangle$, which counts the number of heterochromatic pairs of coincident vertices in G. By definition, G is properly edge-colored if and only if every pair of coincident vertices is heterochromatic. Hence, the result. □

3 Greedy Coloring

As illustrated by Theorems 2.4 and 2.11, the zeon coloring matrices allow one to quickly determine whether or not a given coloring is proper. The task at hand now is to develop a matrix-based greedy coloring algorithm that can be conveniently implemented in *Mathematica* to generate a proper coloring.

The matrix-based algorithm developed here works from right to left across columns of the adjacency matrix, so the vertex ordering is inferred from the construction of the adjacency matrix. To represent a proper vertex coloring, the zeon generators appearing in columns associated with adjacent vertices must be distinct. To this end, Algorithm 1 operates as follows.

The algorithm accepts as input the usual adjacency matrix $A = (a_1|\cdots|a_n)$ and constructs a nilpotent coloring matrix $\Psi = (\psi_1|\cdots|\psi_n)$. After assigning $\Psi \leftarrow A$ as an initialization [2], the algorithm proceeds from left to right.

Considering the jth column ψ_j, let M denote the indices of all zeon generators appearing in ψ_j. Observing that the graph is properly n-colorable, it follows immediately that setting $\chi = \min\{[n] \setminus M\}$ makes ζ_χ the least-indexed generator available to color vertex v_j. This coloring is accomplished by setting $\psi_j = \zeta_\chi a_j$. To make this color unavailable to the remaining uncolored vertices, the algorithm sets $\psi_{ji} = \psi_{ij}$ for $j < i \leq n$. This is repeated as j runs from 1 to n. At the end, the matrix Ψ represents a proper κ-coloring of G, where $\kappa \leq n$ is the maximum index appearing among zeon generators in Ψ.

Example 3.1. A randomly-generated graph on $n = 36$ vertices and $|E| = 237$ edges is seen in Figure 3. This 9-coloring was generated using Algorithm 1. The

[2] Ψ can be initialized as any $n \times n$ matrix. The assignment $\Psi \leftarrow A$ is expedient.

```
input : Adjacency matrix A = (a_1|···|a_n) of a simple graph G on n
        vertices.
output: Proper zeon vertex-coloring matrix Ψ = (ψ_1|···|ψ_n) associated
        with graph G.

Initialize matrix Ψ.;
Ψ ← A;
Begin with first vertex (i.e., first column of Ψ).;
j ← 1;
while j ≤ n do
    Get indices of any ζ's appearing in current column of Ψ.;
    M ← {Indices of generators in ψ_j};
    Choose minimum available color index.;
    χ ← min([n] \ M);
    Set jth column of Ψ to represent color.;
    ψ_j ← ζ_χ a_j ;
    Make this color unavailable to neighbors yet to be evaluated.;
    for i ← j + 1 to n do
        ψ_ji ← ψ_ij;
    end
    j ← j + 1;
end
return Ψ;
```

Algorithm 1: Proper Zeon Vertex Coloring Matrix of a Graph

coloring is proper, as verified by $\langle\langle \mathrm{tr}(\Psi^2)\rangle\rangle = 474 = 2|E|$. In particular,

$$\begin{aligned}\mathrm{tr}(\Psi^2) \quad = \quad & 46\zeta_{\{1,2\}} + 24\zeta_{\{1,3\}} + 16\zeta_{\{1,4\}} + 8\zeta_{\{1,5\}} + 26\zeta_{\{1,6\}} + 10\zeta_{\{1,7\}} \\ & +8\zeta_{\{1,8\}} + 8\zeta_{\{1,9\}} + 42\zeta_{\{2,3\}} + 24\zeta_{\{2,4\}} + 22\zeta_{\{2,5\}} + 32\zeta_{\{2,6\}} \\ & +12\zeta_{\{2,7\}} + 8\zeta_{\{2,8\}} + 2\zeta_{\{2,9\}} + 20\zeta_{\{3,4\}} + 14\zeta_{\{3,5\}} + 22\zeta_{\{3,6\}} \\ & +10\zeta_{\{3,7\}} + 12\zeta_{\{3,8\}} + 6\zeta_{\{3,9\}} + 10\zeta_{\{4,5\}} + 18\zeta_{\{4,6\}} + 6\zeta_{\{4,7\}} \\ & +8\zeta_{\{4,8\}} + 6\zeta_{\{4,9\}} + 12\zeta_{\{5,6\}} + 6\zeta_{\{5,7\}} + 4\zeta_{\{5,8\}} + 4\zeta_{\{5,9\}} \\ & +8\zeta_{\{6,7\}} + 6\zeta_{\{6,8\}} + 2\zeta_{\{6,9\}} + 6\zeta_{\{7,8\}} + 2\zeta_{\{7,9\}} + 4\zeta_{\{8,9\}}.\end{aligned}$$

4 Orthozeons and Monochromatic Walks

While zeons lend themselves nicely to counting heterochromatic self-avoiding walks, some new algebraic tools are required for the monochromatic case.

Given a κ-dimensional vector space V equipped with inner product $\langle \cdot, \cdot \rangle : V \to \mathbb{R}$, it is not difficult to see that for any unit column vector $\mathbf{u} \in V$, the

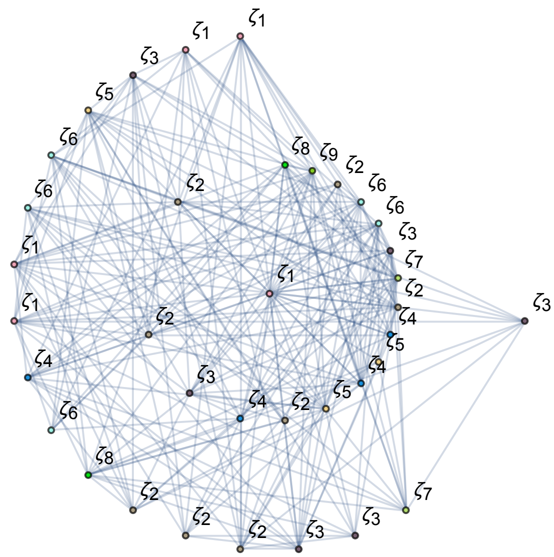

Figure 3: Greedy 9-coloring of a graph on 36 vertices.

outer product $\mathbf{u}\mathbf{u}^\dagger$ is an order-κ matrix that acts on V as orthogonal projection onto span($\mathbf{u}$) via matrix multiplication.

Denoting such a rank-one projection by $\tau_{\mathbf{u}}$, it is also not difficult to see that the product $\tau_{\mathbf{u}}\tau_{\mathbf{v}}$ is the zero matrix when $\langle \mathbf{u}, \mathbf{v}\rangle = 0$. Being a projection, $\tau_{\mathbf{u}}$ is obviously idempotent. Hence, the rank-one projections associated with any orthonormal basis $\{\mathbf{u}_i : 1 \le i \le \kappa\}$ for V generate a commutative κ-dimensional algebra satisfying

$$\tau_{\mathbf{u}_i}\tau_{\mathbf{u}_j} = \begin{cases} 0 & \text{when } i \neq j,\\ \tau_{\mathbf{u}_i} & \text{when } i = j.\end{cases}$$

For notational convenience, the generators will be denoted by $\{\tau_i : 1 \le i \le \kappa\}$. The algebra generated by these projections will be denoted $\mathcal{P}_\kappa$, and it is isomorphic to the algebra of diagonal matrices with real coefficients.

The goal of the current section is to develop methods for counting monochromatic self-avoiding walks in finite graphs. To that end, nilpotent coloring matrices will be defined having entries in the tensor algebra $\mathcal{P}_\kappa \otimes {\mathcal{C}\ell_n}^{\rm nil}$. Generators of this algebra will be referred to as *orthozeons*. For notational covenience, define $\overset{j}{\zeta}_X = \tau_j \otimes \zeta_X$, for $1 \le j \le \kappa$ and $X \subseteq 2^{[n]}$. Orthozeons thereby satisfy the following multiplication rules:

$$\overset{i}{\zeta}_X\overset{j}{\zeta}_Y = \overset{j}{\zeta}_Y\overset{i}{\zeta}_X = \begin{cases} \overset{i}{\zeta}_{X\cup Y} & (i = j) \wedge (X \cap Y = \emptyset),\\ 0 & (i \neq j) \vee (X \cap Y \neq \emptyset).\end{cases} \tag{4.1}$$

Multiplication in the algebra is defined by associative linear extension of the action (4.1) defined on generators. The dimension of the algebra is readily seen to be $\kappa 2^n$.

Constructing an adjacency matrix with orthozeon generators now allows one to count monochromatic self-avoiding walks in colored graphs.

Definition 4.1. Let $G = (V, E)$ be a simple graph on n vertices with vertex coloring $\phi : V \to \{1, \ldots, \kappa\}$. The *orthozeon vertex-coloring matrix* Φ associated with G is the $n \times n$ matrix whose entries are generators of $\mathcal{P}_\kappa \otimes \mathcal{C}\ell_n{}^{\text{nil}}$ defined for $1 \leq i, j \leq n$ by

$$\langle v_i|\Phi|v_j\rangle = \begin{cases} \overset{\phi(j)}{\zeta}_{\{j\}} & \text{if } (v_i, v_j) \in E, \\ 0 & \text{otherwise.} \end{cases}$$

Theorem 4.2. *Let Φ be the orthozeon vertex-coloring matrix of a graph of G on n vertices. Then, for $1 \leq i, j \leq n$ and $m \in \mathbb{N}$,*

$$\langle v_i|\Phi^m|v_j\rangle = \sum_{\ell=1}^{\kappa} \sum_{|I|=m} \alpha_{\ell,I} \overset{\ell}{\zeta}_I$$

where $\alpha_{\ell,I}$ is the number of m-walks from $v_i \to v_j$ in the graph on vertices indexed by I, each of color ℓ, such that no vertex is repeated, with the possible exception of v_i exactly once. In particular, the coefficient of $\overset{\ell}{\zeta}_{I\cup\{i\}}$ in $\overset{\ell}{\zeta}_{\{i\}}\langle v_i|\Phi^m|v_j\rangle$ is the number of monochromatic m-paths of color ℓ from v_i to v_j on vertices indexed by I.

Proof. Proof is by induction on m, using the inherent properties of the algebra $\mathcal{P}_\kappa \otimes \mathcal{C}\ell_n{}^{\text{nil}}$. The structure is similar to the proof of Theorem 2.2. □

Example 4.3. Figure 4 depicts an orthozeon 5-coloring of a graph on 30 vertices. Letting Φ denote an orthozeon coloring matrix of the graph, one finds

$$\begin{aligned} \operatorname{tr}(\Phi^5) \quad = \quad & 10\overset{3}{\zeta}_{\{2,3,19,26,27\}} + 10\overset{3}{\zeta}_{\{2,4,19,26,27\}} + 10\overset{5}{\zeta}_{\{1,5,15,16,22\}} + 10\overset{5}{\zeta}_{\{1,5,15,16,28\}} \\ & +10\overset{5}{\zeta}_{\{1,5,15,22,28\}} + 10\overset{5}{\zeta}_{\{1,5,16,22,28\}}. \end{aligned}$$

Observing that $\langle\langle \operatorname{tr}(\Phi^5)\rangle\rangle = 60$, one concludes that G contains 12 monochromatic 5-cycles. Four of the 5-cycles are on vertices of color 3, and eight are on vertices of color 5. Their respective vertex sets are seen in the subscripts.

Definition 4.4. Let G be a simple (possibly directed) graph with edge κ-coloring θ. Define the *orthozeon edge-coloring matrix* Υ of (G, θ) having entries in $\mathcal{P}_\kappa \otimes \mathcal{C}\ell_n{}^{\text{nil}}$ by

$$\langle v_i|\Upsilon|v_j\rangle = \begin{cases} \overset{\theta(i,j)}{\zeta}_{\{v_i,v_j\}} \text{ if } (v_i, v_j) \in E, \\ 0 \text{ otherwise.} \end{cases}$$

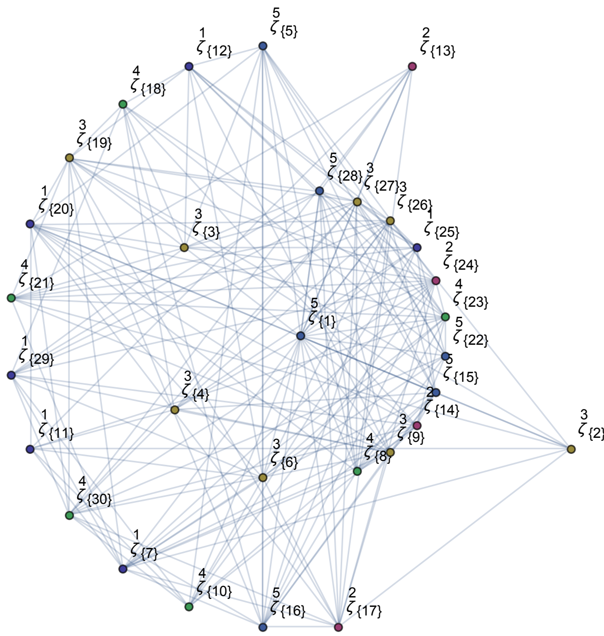

Figure 4: An orthozeon 5-colored 30-vertex graph.

Theorem 4.5. *Let Υ be the orthozeon edge-coloring matrix of (G, θ), where θ is an edge κ-coloring of G. Then, for $1 \le i, j \le n$ and $m \in \mathbb{N}$,*

$$\langle v_i|\Upsilon^m|v_j\rangle = \sum_{\ell=1}^{\kappa} \sum_{|I|=m} \alpha_{\ell,I} \overset{\ell}{\zeta}_I,$$

where $\alpha_{\ell,I}$ denotes the number of monochromatic m-trails $v_i \to v_j$ on edges of color ℓ indexed by I.

Proof. Proof is by induction on m, using the inherent properties of the algebra $\mathcal{P}_\kappa \otimes \mathcal{C}\ell_{[E]}{}^{\text{nil}}$. The structure is similar to the proof of Theorem 2.8. $\square$

5 Zeon & Orthozeon Coloring Polynomials

Traditional graph-theoretic polynomials include chromatic, cycle, and Tutte polynomials. In [14], Harris and Staples define various *spinor polynomials* associated with finite graphs. The spinor polynomials are polynomials in one real variable, t, having spinor-valued coefficients. Depending on the construction, the polynomials reveal the sizes of maximal cliques, independent sets, and matchings. Spinor polynomials can also reveal a graph's girth or circumference.

In the current work, the notion of spinor polynomials is extended to recover information about heterochromatic or monochromatic structures in graphs. By introducing polynomials having zeon or orthozeon coefficients, details of a graph's hetero- or monochromatic cliques, independent sets, matchings, girth, or circumference can be revealed.

Definition 5.1. The *heterochromatic circumference* of a colored graph G is the size of a maximal heterochromatic cycle in G. The *monochromatic circumference* of a colored graph G is the size of a maximal monochromatic cycle in G.

Dual to the notion of circumference is the notion of girth. Hence, the following definition.

Definition 5.2. The *heterochromatic girth* of a colored graph G is the size of a minimal heterochromatic cycle in G. The *monochromatic girth* of a colored graph G is the size of a minimal monochromatic cycle in G.

Two other (dual) structures of interest are independent sets and matchings in graphs. In a graph $G = (V, E)$ a vertex subset $U \subseteq V$ is said to be an *independent set* if no pair of vertices in U is adjacent in the graph; i.e., $u, v \in U$ implies $(u, v) \notin E$ and $(v, u) \notin E$. These definitions extend in the obvious way to define *heterochromatic or monochromatic independent sets.*

Definition 5.3. A *heterochromatic independent set* of a graph $G = (V, E)$ with vertex coloring ϕ is a subset of distinctly-colored vertices of G that are pairwise non-adjacent. More specifically, $U \subseteq V$ is a heterochromatic independent set of G if $u, v \in U$ implies $\phi(u) \neq \phi(v)$, $(u, v) \notin E$, and $(v, u) \notin E$. Similarly, a *monochromatic independent set* of a graph $G = (V, E)$ with vertex coloring ϕ is a single-colored subset of vertices of G that are pairwise non-adjacent. In other words, $U \subseteq V$ is a heterochromatic independent set of G if $u, v \in U$ implies $\phi(u) = \phi(v)$, $(u, v) \notin E$, and $(v, u) \notin E$.

Dual to the notion of an independent set, a *matching* of G is a subset $F \subseteq E$ such that no pair of edges in F is coincident in G. This definition extends naturally to *heterochromatic or monochromatic matchings.*

Definition 5.4. A *heterochromatic matching* of a graph $G = (V, E)$ with edge coloring θ is a subset of distinctly-colored edges of G that are pairwise non-coincident. Equivalently, $F \subseteq E$ is a heterochromatic matching of G if $(a, b), (c, d) \in F$ implies $\theta((a, b)) \neq \theta((c, d))$, and that the sets $\{a, b\}$ and $\{c, d\}$ are disjoint. A *monochromatic matching* of a graph $G = (V, E)$ with edge coloring θ is a single-colored subset of edges of G that are pairwise non-coincident. That is, $F \subseteq E$ is a heterochromatic matching of G if $(a, b), (c, d) \in F$ implies $\theta((a, b)) = \theta((c, d))$, and that the sets $\{a, b\}$ and $\{c, d\}$ are disjoint.

With these concepts in hand, it is now possible to define polynomials that reveal more information about structures contained in colored graphs.

Proposition 5.5. *Let Ψ be a zeon vertex-coloring matrix of G, and define the* zeon coloring polynomial *of G, $\mathfrak{z}(t)$, by*

$$\mathfrak{z}(t) = \operatorname{tr}\left(e^{t\Psi}\right).$$

Then, the coefficient of t^k *in* $\mathfrak{z}(t)$ *is of the form*

$$\langle \mathfrak{z}(t), t^k\rangle = \sum_{|I|=k} k\alpha_I \zeta_I,$$

where α_I *is the number of heterochromatic* k*-cycles in* G *on colors indexed by* I*. In particular, the graph is acyclic if* $\mathfrak{z}(t) = 0$*; otherwise,* $\deg_t(\mathfrak{z}(t))$ *is the heterochromatic circumference of* G*.*

Proof. Given the nilpotent structure of Ψ as a matrix having generators of ${\mathcal{C}\ell_\kappa}^{\rm nil}$ as entries, it is clear that the matrix exponential can be written as a finite sum. Further, by linearity of trace, $\mathfrak{z}(t) = \displaystyle\sum_{\ell=0}^{\kappa} \frac{t^\ell}{\ell!}\mathrm{tr}(\Psi^\ell)$. The result now follows immediately from Corollary 2.3. □

The following corollary is immediate.

Corollary 5.6. *Let* Ψ *be a zeon vertex-coloring matrix of* G *on* n *vertices, and let* $\mathfrak{z}(t)$ *be the zeon coloring polynomial of* G*, as defined in Proposition 5.5. If* $\mathfrak{z}(t) \neq 0$*, then the* heterochromatic girth *of* G *is given by* $n - \deg_t(t^n\mathfrak{z}(1/t))$ *.*

Defining polynomials with orthozeon coefficients allow consideration of a graph's monochromatic subgraphs.

Proposition 5.7. *Let* Φ *be an orthozeon vertex-coloring matrix of* G*, and define the* orthozeon coloring polynomial of G*,* $\mu(t)$*, by*

$$\mu(t) = \mathrm{tr}\left(e^{t\Phi}\right).$$

Then $\deg_t(\mu(t))$ *is the monochromatic circumference of* G*.*

Proof. As a polynomial in t, the degree of $\mathrm{tr}\left(e^{t\Phi}\right)$ is the maximum exponent k for which Φ^k is nonzero. By Theorem 4.2, k is the length of a maximal monochromatic cycle of (G, ϕ). □

The next corollary is immediate.

Corollary 5.8. *Let* Φ *be an orthozeon vertex-coloring matrix of* G *on* n *vertices, and let* $\mu(t)$ *be the orthozeon coloring polynomial of* G*, as defined in Proposition 5.7. If* $\mu(t) \neq 0$*, then the* monochromatic girth *of* G *is given by* $n - \deg_t(t^n\mu(1/t))$ *.*

These results extend naturally to edge-colored graphs in order to reveal details of heterochromatic and monochromatic matchings.

Proposition 5.9. *Let* $G = (V, E)$ *be a simple graph on* n *vertices with edge-coloring* $\phi : E \to \{n+1, \ldots, n+\kappa\}$*. Setting* $\Gamma = \displaystyle\sum_{(v_i,v_j)\in E\subset V\times V} \zeta_{\{v_i,v_j,\phi(v_i,v_j)\}}$*, the exponential* $e^{t\Gamma}$ *is a polynomial in* t *with coefficients in* ${\mathcal{C}\ell_{n+\kappa}}^{\rm nil}$*. Furthermore* $\deg_t(e^{t\Gamma})$ *is the size of a maximal heterochromatic matching in* G*.*

Proof. The key here is that the product $\zeta_{\{v_i,v_j,\phi(v_i,v_j)\}}\zeta_{\{v_\ell,v_m,\phi(v_\ell,v_m)\}}$ is only nonzero if $\{v_i, v_j, v_\ell, v_m, \phi(v_i, v_j), \phi(v_\ell, v_m\}$ is a pairwise-disjoint set. Given that Γ is clearly nilpotent, the exponential is a finite sum (i.e., a polynomial in t), and the degree of t is the maximal number of factors $\zeta_{\{v_i,v_j,\phi(v_i,v_j)\}}$ appearing in any nonzero product taken over all edges in G. For the product to be nonzero, endpoints of the edges are disjoint and colors of the edges are disjoint. Hence, the result. □

Proposition 5.10. *Letting $G = (V, E)$ be a simple graph on n vertices with edge κ-coloring $\phi : E \to \{1, \dots, \kappa\}$. Define $\omega : E \to \mathcal{P}_\kappa \otimes \mathcal{C}\ell_n{}^{\text{nil}}$ by*

$$\omega(v_i, v_j) = \overset{\phi(v_i,v_j)}{\zeta}{}_{\{v_i,v_j\}}.$$

Setting $\Gamma = \sum_{\varepsilon\in E} \omega(\varepsilon)$, the exponential $e^{t\Gamma}$ is seen to be a polynomial in t with orthozeon coefficients such that $\deg_t(e^{t\Gamma})$ is the size of a maximal monochromatic matching in G.

Proof. Nilpotent properties of $\mathcal{P}_\kappa \otimes \mathcal{C}\ell_n{}^{\text{nil}}$ guarantee that the exponential $e^{t\Gamma}$ is a finite sum of the form $e^{t\Gamma} = \sum_{m=0}^{n} \frac{t^m}{m!}\Gamma^m$. Further, for a given m, straightforward application of the multinomial theorem yields the following:

$$\begin{aligned}
\Gamma^m &= \left(\sum_{(v_i,v_j)\in E} \overset{\phi(v_i,v_j)}{\zeta}{}_{\{v_i,v_j\}}\right)^m \\
&= \left(\sum_{\ell=1}^{\kappa} \sum_{\substack{(v_i,v_j)\in E\\ \phi(v_i,v_j)=\ell}} \overset{\ell}{\zeta}{}_{\{v_i,v_j\}}\right)^m \\
&= \left(\sum_{\ell=1}^{\kappa} \sum_{\substack{(v_i,v_j)\in E\\ \phi(v_i,v_j)=\ell}} \tau_\ell \otimes \zeta_{\{v_i,v_j\}}\right)^m \\
&= \sum_{\ell_1+\cdots+\ell_\kappa=m} \binom{m}{\ell_1, \dots, \ell_\kappa} \prod_{q=1}^{\kappa} \left({\tau_q}^{\ell_q} \otimes \left(\sum_{\substack{(v_i,v_j)\in E\\ \phi(v_i,v_j)=q}} \zeta_{\{v_i,v_j\}} \right)^{\ell_q} \right),
\end{aligned}$$

where orthogonality of the τ_q's guarantee that the product taken over q from 1 to κ is nonzero only if $(\kappa-1)$ of the ℓ_q's are zero. Hence, for some $q' \in \{1, \ldots, \kappa\}$,

$$\prod_{q=1}^{\kappa} \left({\tau_q}^{\ell_q} \otimes \left(\sum_{\substack{(v_i,v_j)\in E \\ \phi(v_i,v_j)=q}} \zeta_{\{v_i,v_j\}} \right)^{\ell_q} \right) = {\tau_{q'}}^{\ell_{q'}} \otimes \left(\sum_{\substack{(v_i,v_j)\in E \\ \phi(v_i,v_j)=q'}} \zeta_{\{v_i,v_j\}} \right)^{\ell_{q'}}$$

$$= \tau_{q'} \otimes \left(\sum_{\substack{(v_i,v_j)\in E \\ \phi(v_i,v_j)=q'}} \zeta_{\{v_i,v_j\}} \right)^{\ell_{q'}}.$$

The multinomial theorem now further implies that $\left(\sum_{\substack{(v_i,v_j)\in E \\ \phi(v_i,v_j)=q'}} \zeta_{\{v_i,v_j\}} \right)^{\ell_{q'}}$ is nonzero if and only if there exists a matching of size ℓ_q (i.e., a collection of ℓ_q edges whose endpoints form a pairwise disjoint collection) in the graph. Further, the edges of this matching are monochromatic of color q'. The largest exponent $\ell_{q'}$ for which the expression is nonzero is thereby the size of a maximal monochromatic matching in the graph. One sees immediately that this maximal exponent is the degree of $e^{t\Gamma}$ as a polynomial in t. □

6 Concluding Remarks

While graph colorings have been studied for many years, reformulating graph coloring problems within a new algebraic framework opens up new avenues of discovery and offers new opportunities for interdisciplinary research. In particular, these methods could be applied to Boolean satisfiability problems, as their connections with graph colorings are well known and have been explored in numerous works. The Boolean satisfiability problem, or SAT, is the problem of determining whether the variables of a given Boolean formula can be consistently replaced by true or false in such a way that the formula evaluates to true. In fact, SAT was the first known NP-complete problem [7].

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