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Some aspects of topological groups

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I. INTRODUCTION

In this introduction, we list the basic concepts and the basic theorems which are used in the main body of the paper. These theorems are available in most of the standard textbooks on the subject, such as those given in the List of References, though the proofs of some of the simpler ones are not given and have, therefore, been included here. All lemmas in the succeeding chapters are also of this nature. However, to the extent that the author was unable to find them in the literature, the theorems in Chapters II through IV are original. The definitions in these chapters also are original.

DEFINITION. A <u>topological space</u> $(X,\mathcal{J})$ is a system consisting of a set X and a collection $\mathcal{J}$ of subsets of X, called <u>open sets</u>, $\ni$ the following hold:

$$(1)\ X = \bigcup_{U \in \mathcal{J}} U,$$

$$(2)\ T \subset \mathcal{J} \Rightarrow \bigcup_{U \in T} U \in \mathcal{J},$$

$$(3)\ U_1, U_2 \in \mathcal{J} \Rightarrow U_1 \cap U_2 \in \mathcal{J}.$$

If $(X,\mathcal{J})$ is a topological space, $\mathcal{J}$ is said to be a <u>topology</u> for X.

DEFINITION. Let $(X,\mathcal{J})$ be a topological space and let $x \in X$. A subset W of X is said to be a <u>neighborhood</u> (nbd) of x if $\exists\ U \in \mathcal{J} \ni x \in U \subset W$.

DEFINITION. Let $(X,\mathcal{T})$ be a topological space. Then $\mathcal{B} \subset \mathcal{T}$ is a base for $\mathcal{T}$ <=>, $\forall x \in X$ and $\forall$ nbd U of x, $\exists B \in \mathcal{B} \ni x \in B \subset U$.

DEFINITION. Let $(X,\mathcal{T})$ and $(X', \mathcal{T}')$ be topological spaces. A mapping f of X into X' is continuous with respect to $\mathcal{T}$ and $\mathcal{T}'$ <=> $f^{-1}[U'] = \{u \in X | f(u) \in U'\} \in \mathcal{T}$.

DEFINITION. Let $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ be topological spaces. A mapping f of X onto X' is a homeomorphism of $(X,\mathcal{T})$ and $(X',\mathcal{T}')$ <=> f is 1-1 and f and f^{-1} are continuous.

DEFINITION. A topological group is a system $(G,\cdot,\mathcal{T}) \ni (G,\cdot)$ is a group, $(G,\mathcal{T})$ is a topological space, and $\forall x, y \in G$ and $\forall$ nbd W of xy^{-1}, $\exists$ nbds U and V of x and y, respectively, $\ni UV^{-1} = \{u \cdot v^{-1} | u \in U, v \in V\} \subset W$.

All groups (and topological groups) will be annotated multiplicatively, with juxtaposition often used to indicate the group operation. Hence, no distinction will be made between (possibly unlike) operations. Frequently, mention of group operations and/or topologies will be omitted and "G" will be used to denote a group $(G,\cdot)$ or a topological group $(G,\cdot,\mathcal{T})$.

DEFINITION. Let $(G,\cdot)$ and $(G',\cdot)$ be groups. A mapping α of G into $G' \ni \alpha(xy) = \alpha(x)\alpha(y)$, $\forall x, y \in G$, is called a homomorphism. If, in addition, α is a 1-1 mapping of G onto G', then α is said to be an isomorphism of $(G,\cdot)$ and $(G',\cdot)$. A homomorphism (isomorphism) of a group into (onto) itself is called an endomorphism (automorphism).

DEFINITION. A <u>group with operators</u> is a system $(G,\cdot,\Phi) \ni (G,\cdot)$ is a group and Φ is a set of endomorphisms of $(G,\cdot)$. We will also use "G" to denote a group with operators. Two groups G and G' with the same set of operators, Φ, are said to be <u>isomorphic</u> if $\exists$ an isomorphism $f\colon G \to G' \ni \forall \alpha \in \Phi$ and $\forall a \in G$, $f(\alpha(a)) = \alpha(f(a))$.

DEFINITION. Let $(G,\cdot,\Phi)$ be a group with operators. If $G_0, G_1,\ldots,G_n$ is a (finite) sequence of subgroups of $(G,\cdot) \ni$ (1) $\forall \alpha \in \Phi$, $\alpha(G_i) \subset G_i$, $0 \leq i \leq n$, (2) $G_0 = G$, (3) $G_n = \{e\}$, where e is the neutral element of G, and (4) G_i is an invariant subgroup of G_{i-1}, $1 \leq i \leq n$, then $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ is called a <u>composition series</u> for G. If Σ and Σ' are composition series for $G \ni$ every term of Σ' is a term of Σ, then Σ' is said to be a <u>refinement</u> of Σ. Two composition series are said to be <u>equivalent</u> if $\exists$ a 1-1 correspondence between the quotient groups of the two series $\ni$ corresponding quotient groups are isomorphic.

DEFINITION. A <u>Jordan-Hölder series</u> for a group with operators $(G,\cdot,\Phi)$ is a composition series $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\} \ni \forall i \in \{1,\ldots,n\}$, G_i is maximal in G_{i-1}, i.e., if H is an invariant subgroup of $G_{i-1} \ni \alpha(H) \subset H$, $\forall \alpha \in \Phi$, and $\ni$ if $H \supset G_i$, then either $H = G_{i-1}$ or $H = G_i$.

DEFINITION. Let $(X,\mathcal{J})$ be a topological space and let R be an equivalence relation on X. Then the decomposition X/R of X into equivalence classes together with the topology $\mathcal{J}' = \{\mathcal{Q} \subset X/R \mid \bigcup_{A \in \mathcal{Q}} A \in \mathcal{J}\}$ is said to be a quotient space and $\mathcal{J}'$ is called the quotient topology.

DEFINITION. A topological group with operators is a system $(G,\cdot,\mathcal{J},\Phi) \ni (G,\cdot,\mathcal{J})$ is a topological group and Φ is a set of endomorphisms of $(G,\cdot) \ni \forall \alpha \in \Phi$, α is continuous.

DEFINITION. A topological space is said to be regular if for each point x of the space and for each nbd U of x, there is a closed nbd V of x such that $V \subset U$. A topological space is said to be normal if for each pair of disjoint closed sets, A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

DEFINITION. There is a well-known finite sequence, $T_0,\ldots,T_4$, of separation axioms which a topological space $(X,\mathcal{J})$ may satisfy (see, for example, Kelley [4]). If $(X,\mathcal{J})$ satisfies the axiom T_i, $0 \leq i \leq 4$, then $(X,\mathcal{J})$ is said to be a T_i-space.

The following theorem is a well-known elementary result in the theory of topological groups (see, for example, [6,p.53]).

THEOREM 1. Let $(G,\cdot,\mathcal{J})$ be a topological group and let $x \in G$. Then the mappings L_x and R_x of G onto G $\ni \forall a \in G$, $L_x(a) =$

xa, $R_x(a) = ax$, called, respectively, the left and right translations by x, are homeomorphisms.

THEOREM 2. Let $(G,\cdot,\mathcal{J})$ be a topological group and let e be the neutral element of $(G,\cdot)$. Then $U \in \mathcal{J} \Rightarrow x^{-1}U$ is an open nbd of e, $\forall\, x \in U$.

__Proof__: Let $U \in \mathcal{J}$ and let $x \in U$. Then $x^{-1}x = e \in x^{-1}U$. And, since the left translation by x^{-1} is a homeomorphism, by the above theorem, $x^{-1}U \in \mathcal{J}$. Therefore, $x^{-1}U$ is an open nbd of e. ||

THEOREM 3. If f is an isomorphism of a topological group $(G,\cdot,\mathcal{J})$ onto a topological group $(G',\cdot,\mathcal{J}')$ ∋ the inverse of any nbd of the neutral element e' of G' is a nbd of the neutral element e of G, then f is continuous.

__Proof__: Suppose f is not continuous. Then $\exists\, U' \in \mathcal{J}' \ni f^{-1}[U'] \notin \mathcal{J}$. Let $x \in f^{-1}[U']$. Then $\exists\, y \in U' \ni f(x) = y$. By the above theorem, $y^{-1}U'$ is an open nbd of e'. Let $z = f^{-1}(y^{-1})$. Since f is an isomorphism, $f^{-1}[y^{-1}U'] = zf^{-1}[U']$, which $\Rightarrow z^{-1}f^{-1}[y^{-1}U'] = f^{-1}[U']$. Suppose $f^{-1}[y^{-1}U'] \in \mathcal{J}$. Then, since the left translation by z^{-1} is a homeomorphism (Theorem 1), $f^{-1}[U'] \in \mathcal{J}$, contrary to hypothesis.

Hence, f not continuous $\Rightarrow \exists$ a nbd of e' ∋ the inverse of that nbd is not a nbd of e. ||

The following theorem gives a well-known property of the nbds of the neutral element in a topological group (see, for example, Pontrjagin [6,p.55]).

THEOREM 4. **If U is a nbd of the neutral element e in a topological group $(X,\cdot,\mathcal{J})$, then $\exists$ a nbd V of e $\ni V^{-1}V \subset U$.**

THEOREM 5. **A topological group is T_2 whenever it is T_0.**

Proof: Let $(X,\cdot,\mathcal{J})$ be a topological group $\ni (X,\mathcal{J})$ is a T_0-space. Let x, y $\in$ X. Then either $\exists$ a nbd of x to which y does not belong or $\exists$ a nbd of y to which x does not belong. For definiteness, assume that the latter holds. Then $\exists$ a nbd U of y $\ni x \notin U$. By Theorem 2, Uy^{-1} is a nbd of the neutral element e of X. Then, by the above theorem, $\exists$ a nbd V of e $\ni V^{-1}V \subset W$. Since the right translations R_x and R_y by x and y, respectively, are homeomorphisms, $R_x[V] = Vx$ and $R_y[V] = Vy$ are nbds of x and y, respectively.

Suppose $\exists\, z \in X \ni z \in Vx \cap Vy$. Then $\exists\, v, v' \in V \ni z = vx = v'y$, which $\Rightarrow zx^{-1}, zy^{-1} \in V$. Hence, $(zx^{-1})^{-1}(zy^{-1}) = (xz^{-1})(zy^{-1}) = x(z^{-1}z)y^{-1} = xy^{-1} \in V^{-1}V \subset Wy^{-1}$. However, $x \notin W \Rightarrow xy^{-1} \notin Wy^{-1}$. Hence, $Vx \cap Vy = \emptyset$.

Therefore, $(X,\cdot,\mathcal{J})$ is a T_2-space. ||

The following theorem is an extension of the concept of a quotient group of a group to a group with operators. The proof is straightforward (see, for example, Jacobson [3,p.131]).

THEOREM 6. Let $(G,\cdot,\Phi)$ be a group with operators and let H be an invariant subgroup of G. $\forall\ \alpha \varepsilon \Phi$ and $\forall\ \bar{a} = aH \varepsilon G/H$, where $a \varepsilon \bar{a} \subset G$, define $\alpha(\bar{a}) = \alpha(a)H$. Then $(G/H,\cdot,\Phi)$ is a group with operators.

The following four theorems are well-known. For the proofs, see Bourbaki [1,pp.85-87].

THEOREM 7. (Schreier) If Σ_1 and Σ_2 are two composition series for a group with operators G, then $\exists$ refinements Σ_1' and Σ_2' of Σ_1 and Σ_2, respectively, $\ni$ Σ_1' and Σ_2' are equivalent.

THEOREM 8. (Zassenhaus) Let $(G,\cdot,\Phi)$ be a group with operators and let H and K be invariant subgroups of G $\ni$ $\forall\ \alpha \varepsilon \Phi$, $\alpha[H] \subset H$ and $\alpha[K] \subset K$. Then, if H' and K' are invariant subgroups of H and K, respectively, $\ni$ $\forall\ \alpha \varepsilon \Phi$, $\alpha[H'] \subset H'$ and $\alpha[K'] \subset K'$, the following hold:

(1) $H'(H \cap K')$ is an invariant subgroup of $H'(H \cap K)$,

(2) $K'(K \cap H')$ is an invariant subgroup of $K'(K \cap H)$,

(3) the quotient groups $(H'(H \cap K))/(H'(H \cap K'))$ and $(K'(K \cap H))/(K'(K \cap H'))$ are isomorphic.

THEOREM 9. (Jordan-Hölder) Any two Jordan-Hölder series for the same group with operators are equivalent.

THEOREM 10. Let $(G,\cdot,\Phi)$ be a group with operators and let Σ be a Jordan-Hölder series for G. Then if G_i/G_{i+1} is any quotient group of Σ, G_i/G_{i+1} is simple, in the sense that if A is any invariant subgroup of $G_i/G_{i+1} \ni \alpha[A] \subset A$, $\forall \alpha \in \Phi$, then either A = $\{\bar{e}\}$, where $\bar{e}$ is the neutral element of G_i/G_{i+1} or A = G_i/G_{i+1}.

Since a subset G' of a group G "inherits" the property of associativity from G, a nonempty subset G' of G is a subgroup of G <=> the following hold:

(1) $\forall$ a, b $\in$ G', ab $\in$ G',

(2) $\forall$ a $\in$ G', $a^{-1} \in$ G',

(3) e $\in$ G', where e is the neutral element of G.

An equivalent condition is given in the following theorem.

THEOREM 11. A nonempty subset G' of a group G is a subgroup of G <=> $\forall$ a, b $\in$ G', $ab^{-1} \in$ G'.

<u>Proof</u>: Let G be a group and let e be the neutral element of G. Suppose G' is a subgroup of G. Let a, b $\in$ G'. Then, a, b^{-1} $\in$ G', which => $ab^{-1} \in$ G'.

Suppose that $\forall$ a, b $\in$ G', $ab^{-1} \in$ G'. Let a $\in$ G'. Then aa^{-1} = e $\in$ G'. Hence, $ea^{-1} = a^{-1} \in$ G. Let a, b $\in$ G'. Then a, $b^{-1} \in$ G', which => $a(b^{-1})^{-1}$ = ab $\in$ G'. ||

THEOREM 12. Let G be a group and let H be a subgroup of G. Then $HH = HH^{-1} = H$.

Proof: Let H be a subgroup of the group G and let e be the neutral element of G. Let $h \in H$. Then $h = he \in HH$. And, since $e^{-1} = e$, $h = he = he^{-1} \in HH^{-1}$. Hence, $H \subset HH$ and $H \subset HH^{-1}$.

Let $h' \in HH$. Then $\exists\, h_1, h_2 \in H \ni h' = h_1h_2$. Since H is a group, $h_1, h_2 \in H \Rightarrow h_1h_2 \in H$. Hence, $H \supset HH$. Let $h' \in HH^{-1}$. Then $\exists\, h_1, h_2 \in H \ni h' = h_1h_2$. By the above theorem, $h_1, h_2 \in H \Rightarrow h_1, h_2^{-1} \in H$. Hence, $H \supset HH^{-1}$.

Therefore, $HH = HH^{-1} = H$. ||

The two statements in the following theorem are elementary results which can be found in any of the basic texts on topology and algebra, respectively.

THEOREM 13. If α: $X \to X'$ and β: $X' \to X''$, where X, X', and X'' are topological spaces, are homeomorphisms, then α^{-1} is a homeomorphism and the composite $\alpha\beta$ is a homeomorphism. An analogous result holds for isomorphisms of groups.

The following is a well-known result of group theory (see, for example, Lindstrum [5,p.61]).

THEOREM 14. Let G be a group. Then the set of all automorphisms of G is a group and the set of all inner automorphisms of G is an invariant subgroup of this group.

The following result can be found in Kelley [4,p.47].

THEOREM 15. A collection $\mathcal{B}$ of subsets of a set X is a base for a topology for X $\Leftrightarrow$ $X = \bigcup_{B \in \mathcal{B}} B$ and $\forall$ B, B' $\in \mathcal{B}$ and $\forall$ x $\in B \cap B'$, $\exists$ B'' $\in \mathcal{B} \ni x \in B'' \subset U \cap V$.

II. A JORDAN-HÖLDER THEOREM FOR TOPOLOGICAL GROUPS

Let $(G,\cdot,\Phi)$ be a group with operators, $\mathcal{J}$ a topology for G $\ni$ $(G,\cdot,\mathcal{J},\Phi)$ is a topological group with operators, and $G = G_0 \supset G_1 \supset \cdots \supset G_n = \{e\}$ and $G = H_0 \supset H_1 \supset \cdots \supset H_m = \{e\}$, where e is the neutral element of G, be two composition series for $(G,\cdot)$. Then the G_i, $1 \leq i \leq n$, and the H_j, $1 \leq j \leq m$, are topological groups with the relative topologies and the quotient groups G_i/G_{i+1}, $0 \leq i \leq n-1$, and H_j/H_{j+1}, $0 \leq j \leq m-1$, are topological groups with the quotient topologies (see [2,p.71]). The terms and the quotient groups of the series are also groups with operators (with the set of operators Φ), by definition of composition series and Theorem I.6.

DEFINITION. We define two composition series to be <u>topologically equivalent</u> <=> $\exists$ a 1-1 correspondence between the quotient groups of the two series $\ni$ corresponding quotient groups are (1) isomorphic groups with operators and (2) homeomorphic topological spaces.

THEOREM 1. Topologically equivalent composition series are equivalent.

<u>Proof</u>: This follows immediately from the definition of equivalent composition series. ||

Let $(G,\cdot,\mathcal{J},\Phi)$ be a topological group with operators and let $\Sigma_1 = (G_i)_{0\leq i\leq n}$ and $\Sigma_2 = (H_j)_{0\leq j\leq n}$ be any two Jordan-Hölder series for G.

Since Jordan-Hölder series are composition series, we have, by Schreier's Theorem (Theorem I.7) that $\Sigma_1' = (G_{ij})_{\substack{0\leq i\leq n-1\\0\leq j\leq n}}$ and $\Sigma_2' = (H_{ji})_{\substack{0\leq j\leq n-1\\0\leq i\leq n}}$ where $G_{ij} = (G_i \cap H_j)G_{i+1}$, $0\leq i\leq n-1$, $0\leq j\leq n$, and $H_{ji} = (G_i \cap H_j)H_{j+1}$, $0\leq j\leq n-1$, $0\leq i\leq n$, are composition series which are (equivalent) refinements of Σ_1 and Σ_2, respectively. Furthermore, it is clear from the definition that a Jordan-Hölder series has no proper refinements. Hence, $(G_{ij} \in \Sigma_1' \iff G_{ij} \in \Sigma_1)$ and $(H_{ji} \in \Sigma_2' \iff H_{ji} \in \Sigma_2)$.

Now, $\forall$ i, j $\in \{0,1,\ldots,n-1\}$, $G_{ij}/G_{i,j+1} = (G_i \cap H_j)G_{i+1}/(G_i \cap H_{j+1})G_{i+1}$ is isomorphic to $H_{ji}/H_{j,i+1} = (G_i \cap H_j)H_{j+1}/(G_{i+1} \cap H_j)H_{j+1}$ (see Theorem I.8).

Since the Jordan-Hölder Theorem requires only the existence of this set of isomorphisms, the mappings are not given explicitly in the usual proof of the theorem. (See, for example, [1,p.87] and [3,p.141].) We therefore prove the following lemma.

LEMMA 2. Let $\Sigma_1' = (G_{ij})_{\substack{0\le i\le n-1\\ 0\le j\le n}}$ and $\Sigma_2' = (H_{ji})_{\substack{0\le j\le n-1\\ 0\le i\le n}}$ be the series described above. Then, $\forall\ i, j \in \{0,1,\ldots,n-1\}$, the mapping $f_{ij}\colon\ G_{ij}/G_{i,j+1} \to H_{ji}/H_{j,i+1} \ni \forall\ \bar{x} = x(G_i \cap H_{j+1})G_{i+1} \in G_{ij}/G_{i,j+1}$, where $x \in G_i \cap H_j$, $f_{ij}(\bar{x}) = x(G_{i+1} \cap H_j)H_{j+1}$ is an isomorphism.

<u>Proof</u>: Let $i, j \in \{0,1,\ldots,n-1\}$ and let $f = f_{ij}$. We first note that since $G_i \cap H_{j+1} \subset G_i$ and G_{i+1} is an invariant subgroup of G_i, $(G_i \cap H_{j+1})G_{i+1} = G_{i+1}(G_i \cap H_{j+1})$. Let $y \in (G_i \cap H_j)G_{i+1}$. Then $\exists$ $x \in G_i \cap H_j$ and $g_{i+1} \in G_{i+1} \ni y = xg_{i+1}$. Then $y(G_i \cap H_{j+1})G_{i+1} = yG_{i+1}(G_i \cap H_{j+1}) = xg_{i+1}G_{i+1}(G_i \cap H_{j+1})$. But, G_{i+1} a group and $g_{i+1} \in G_{i+1} \Rightarrow g_{i+1}\,G_{i+1} = G_{i+1}$. So, $y(G_i \cap H_{j+1})G_{i+1} = xG_{i+1}(G_i \cap H_{j+1}) = x(G_i \cap H_{j+1})G_{i+1}$. Hence, $\forall\ y \in (G_i \cap H_j)G_{i+1}$, $\exists\ x \in G_i \cap H_j \ni y(G_i \cap H_{j+1})G_{i+1} = x(G_i \cap H_{j+1})G_{i+1}$, which $\Rightarrow G_{ij}/G_{i,j+1}$ is the domain of f.

Let $\bar{z} = z(G_{i+1} \cap H_j)H_{j+1} \in H_{ji}/H_{j\ j+1}$. Then $\exists\ x \in G_i \cap H_j$ and $h_{j+1} \in H_{j+1} \ni z = xh_{j+1}$, which $\Rightarrow \bar{z} = xh_{j+1}(G_{i+1} \cap H_j)H_{j+1}$. Since

$G_{i+1} \cap H_j \subset H_j$ and H_{j+1} is an invariant subgroup of H_j, $(G_{i+1} \cap H_j) \cdot H_{j+1} = H_{j+1}(G_{i+1} \cap H_j)$. Hence, $\bar{z} = xh_{j+1}H_{j+1}(G_{i+1} \cap H_j)$. And, since H_{j+1} is a group, $h_{j+1}H_{j+1} = H_{j+1}$, which => $\bar{z} = xH_{j+1}(G_{i+1} \cap H_j) = x(G_{i+1} \cap H_j)H_{j+1} = f(\bar{x})$, where $\bar{x} = x(G_i \cap H_{j+1})G_{i+1}$. Therefore, f is a mapping of $G_{ij}/G_{i,j+1}$ onto $H_{ji}/H_{j,i+1}$.

Let $\bar{x}_1 = x_1 (G_i \cap H_{j+1})G_{i+1}$, $\bar{x}_2 = x_2(G_i \cap H_{j+1})G_{i+1} \in G_{ij}/G_{i,j+1}$, where $x_1, x_2 \in G_i \cap H_j$, $\ni$ $f(\bar{x}_1) = f(\bar{x}_2)$. Then $x_1(G_{i+1} \cap H_j)H_{j+1} = x_2(G_{i+1} \cap H_j)H_{j+1}$. Since (1) H_{j+1} is an invariant subgroup of H_j, (2) $x_1, x_2 \in G_i \cap H_j \subset H_j$, and (3) $G_{i+1} \cap H_j \subset H_j$, we have that $x_1(G_{i+1} \cap H_j)H_{j+1} = x_2(G_{i+1} \cap H_j)H_{j+1}$ <=> $x_1(G_{i+1} \cap H_j) = x_2(G_{i+1} \cap H_j)$. Since $G_{i+1} \cap H_j \subset G_{i+1}$ and $x_1, x_2 \in G_i \cap H_j \subset G_i$, which is the group of which G_{i+1} is an invariant subgroup, $x_1(G_{i+1} \cap H_j) = x_2(G_{i+1} \cap H_j)$ <=> $x_1 = x_2$. Hence, $\bar{x}_1 = \bar{x}_2$. Therefore, f is 1-1.

Let $\bar{x}_1 = x_1(G_i \cap H_{j+1})G_{i+1}$, $\bar{x}_2 = x_2(G_i \cap H_{j+1})G_{i+1} \in G_{ij}/G_{i,j+1}$. Then, by definition of multiplication of cosets, $\bar{x}_1\bar{x}_2 = x_1x_2(G_i \cap H_{j+1})G_{i+1}$. And, $f(\bar{x}_1)f(\bar{x}_2) = (x_1(G_{i+1} \cap H_j)H_{j+1}) \cdot (x_2(G_{i+1} \cap H_j)H_{j+1}) = x_1x_2(G_{i+1} \cap H_j)H_{j+1} = f(\bar{x}_1\bar{x}_2)$. Therefore, f "preserves" the group operation.

Finally, $\forall$ i, j ε {0,1,...,n-1} and $\forall \alpha \varepsilon \Phi$, $f_{ij}(\alpha(\bar{x}))$ =

$\alpha(x)(G_{i+1} \cap H_j)H_{j+1}$ = $(f_{ij}(\bar{x}))$, $\forall \bar{x} \varepsilon G_i \cap H_j$.

Therefore, $\forall$ i, j ε {0,1,...,n-1}, f_{ij} is an isomorphism. ||

THEOREM 3. Let $(G,\cdot,\mathcal{J},\Phi)$ be a topological group. Then any two Jordan-Hölder series for G are topologically equivalent.

<u>Proof</u>: Let $\Sigma_1 = (G_i)_{0 \leq i \leq n}$ and $\Sigma_2 = (H_j)_{0 \leq j \leq n}$ be any two Jordan-Hölder series for G and let Σ_1' and Σ_2' be the (equivalent) refinements of Σ_1 and Σ_2, respectively, described in the discussion preceding Lemma 2, above. Clearly, we need only show that the mappings f_{ij} defined in Lemma 2 are homeomorphisms.

Let i, j ε {0,1,...,n-1} and let f = f_{ij}. Then $f^{-1}: H_{ji}/H_{j,j+1} \rightarrow G_{ij}/G_{i,j+1} \ni \forall \bar{x} = xH_{j,i+1} \varepsilon H_{ji}/H_{j,i+1}$, where x ε $(G_i \cap H_j)$, $f^{-1}(\bar{x}) = xG_{i,j+1}$.

Let $U \subset H_{ji}/H_{j,j+1}$ be a nbd of the neutral element in $H_{ji}/H_{j,i+1}$. Then, by definition of nbd in the quotient space, $\exists$ $U' \subset G \ni U'$ is a nbd of the neutral element e of G and $\ni U = (U' \cap H_{ji})H_{j,i+1}$. By the definition of Σ_2', the fact that H_{j+1} is an invariant subgroup of H_j which contains $G_{i+1} \cap H_j$, the distributive property of intersections, and the fact that $H_{j+1}H_{j+1} = H_{j+1}$ since H_{j+1} is a group, we have

$$U = (U' \cap H_{ji})H_{j,j+1}$$

$$= (U' \cap (G_i \cap H_j)H_{j+1})(G_{i+1} \cap H_j)H_{j+1}$$

$$= (U' \cap (G_i \cap H_j)H_{j+1})H_{j+1}(G_{i+1} \cap H_j)$$

$$= (U'H_{j+1}(G_i \cap H_j)H_{j+1}H_{j+1})(G_{i+1} \cap H_j)$$

$$= (U'H_{j+1} \cap (G_i \cap H_j)H_{j+1})(G_{i+1} \cap H_j)$$

$$= (U' \cap (G_i \cap H_j))H_{j+1}(G_{i+1} \cap H_j)$$

$$= (U' \cap (G_i \cap H_j))H_{j,i+1}$$

And, $f^{-1}[U] = f^{-1}[(U' \cap (G_i \cap H_j))H_{j,i+1}] = (U' \cap (G_i \cap H_j))G_{i,j+1}$ $= (U' \cap G_{ij})G_{i,j+1}$ (as is apparent by analogy with the foregoing), which is a nbd of the neutral element in $G_{ij}/G_{i,j+1}$. Therefore, f is continuous (see Theorem I.3).

The proof that f^{-1} is continuous is entirely analogous to the proof that f is continuous. Hence, f is a homeomorphism of $G_{ij}/G_{i,j+1}$ onto $H_{ji}/H_{j,i+1}$.

Therefore, $\forall$ i, j ε {0,1,...,n-1}, f_{ij} is a homeomorphism. ||

III. TOPOLOGICAL GROUP GENERATED BY AN INVARIANT SUBGROUP

THEOREM 1. Let $(G,\cdot)$ be any group and let $(H,\cdot)$ be any invariant subgroup of $(G,\cdot)$. Let $T = \{aH | a \in G\}$ and let $\mathcal{T} = \{\bigcup_{A \in T'} A | T' \subset T\}$. Then $(G,\cdot, \ )$ is a topological group.

<u>Proof</u>: Clearly, the union of the members of any subfamily of $\mathcal{T}$ is a member of $\mathcal{T}$ and $G = \bigcup_{U \in \mathcal{T}} U$. And, since any two distinct members of T are disjoint, $A_1, A_2 \in \mathcal{T} \Rightarrow$ either $A_1 \cap A_2 = \emptyset$ or A_1 and A_2 are unions of members of $\mathcal{T}$ and $A_1 \cap A_2$ is a member of T or a union of members of T. Hence, $A_1, A_2 \in \mathcal{T} \Rightarrow A_1 \cap A_2 \in \mathcal{T}$. Thus, $\mathcal{T}$ is a topology for G.

Let $x, y \in G$ and let W be a nbd of xy^{-1} in $(G,\mathcal{T})$. Then xH and yH are nbds of x and y, respectively. And, $\exists\ W' \in \mathcal{T} \ni W' \subset W$ and $xy^{-1} \in W'$. By definition of $\mathcal{T}$, any member of $\mathcal{T}$ containing xy^{-1} must contain the unique member $xy^{-1}H$ of T of which xy^{-1} is a member. Hence, $xy^{-1}H \subset W' \subset W$. But, $xy^{-1}H = xHy^{-1} = xHHy^{-1} = xHH^{-1}y^{-1} = (xH)(yH)^{-1}$, by Theorem I.12.

Therefore, $(G,\cdot,\mathcal{T})$ is a topological group. ||

DEFINITION. The topology $\mathcal{T}$ described in the above theorem will be called the <u>topology generated by the invariant subgroup H</u> of G.

THEOREM 2. Let $(G,\cdot,\mathcal{J})$ be a topological group. Then $\mathcal{J}$ is the topology generated by some invariant subgroup H of G <=> $\forall$ T ε $\mathcal{J} \ni$ T $\neq \emptyset$, $\exists$ W ε $\mathcal{J} \ni$ W $\neq \emptyset$, W $\subset$ T, and no nonempty member of $\mathcal{J}$ is a proper subset of W.

<u>Proof</u>: If $\mathcal{J}$ is the topology generated by an invariant subgroup H of G, then $\forall$ T ε $\mathcal{J} \ni$ T $\neq \emptyset$, $\exists$ a ε G $\ni$ aH $\subset$ T. Clearly, aH ε $\mathcal{J}$, aH $\neq \emptyset$, and no nonempty member of $\mathcal{J}$ is a proper subset of aH.

Conversely, suppose that $\forall$ T ε $\mathcal{J} \ni$ T $\neq \emptyset$, $\exists$ W ε $\mathcal{J} \ni$ W $\neq \emptyset$, W $\subset$ T, and no nonempty member of $\mathcal{J}$ is a proper subset of W. Let $\mathcal{W}$ = {W ε $\mathcal{J}$ | W $\neq \emptyset$ and $\nexists$ T ε $\mathcal{J} \ni$ T $\neq \emptyset$ and T is a proper subset of W} and let e be the neutral element of G.

We will first show that $\exists$ a unique member W' of $\mathcal{W} \ni$ e ε W'. Let W ε $\mathcal{W}$ and let x ε W. Then W ε $\mathcal{J}$, which => $x^{-1}W$ is a nbd of e (see Theorem I.2), which => $\exists$ V ε $\mathcal{J} \ni$ e ε V and V $\subset x^{-1}W$. Let W' ε $\mathcal{W} \ni$ W' $\subset$ V. Since the left translation by x is a homeomorphism (see Theorem I.1) and W' ε $\mathcal{W}$, xW' ε $\mathcal{J}$. Since W' $\subset$ V $\subset x^{-1}W$, xW' $\subset$ W. Hence, by definition of $\mathcal{W}$, xW' = W, which => W' = $x^{-1}W$, which => e ε W'. Suppose $\exists$ W'' ε $\mathcal{W} \ni$ e ε W''. Then W' $\cap$ W'' ε $\mathcal{J}$ and {e} $\subset$ W' $\cap$ W'' $\subset$ W'. Hence by definition of $\mathcal{W}$, W' $\cap$ W'' = W'. Similarly, since W' $\cap$ W'' $\subset$ W'', W' $\cap$ W'' = W''. Hence, $\exists$ a unique member W' of $\mathcal{W} \ni$ e ε W'.

Since W' is a nbd of e, $\exists$ a nbd V of e $\ni VV^{-1} \subset$ W (see Theorem I.4). Hence, $\exists$ U ε $\mathcal{J} \ni$ e ε U $\subset$ V, which => $UU^{-1} \subset VV^{-1} \subset$ W'. But, e ε U^{-1} => U $\subset UU^{-1}$. Hence, since W' ε $\mathcal{W}$, U = W', which => U = UU^{-1} = W'. So, w_1, w_2 ε W' => w_1, w_2 ε U, which =>

$w_1 w_2^{-1} \varepsilon UU^{-1} = W'$. Therefore, W' is a subgroup of G (see Theorem I.11).

Let $x \varepsilon G$. Since the left and right translations by x are homeomorphisms, xW', $W'x \varepsilon \mathcal{J}$. Hence, $\exists\ W \varepsilon \mathcal{W} \ni W \subset xW'$. Since the left translation by x^{-1} is a homeomorphism, $x^{-1}W \varepsilon \mathcal{J}$. And, $W \subset xW'$ $\Rightarrow x^{-1}W \subset W'$, which $\Rightarrow x^{-1}W = W'$, which $\Rightarrow W = xW'$. Hence, $xW' \varepsilon \mathcal{W}$. Similarly, $W'x \varepsilon \mathcal{W}$. Since $e \varepsilon W' \Rightarrow x \varepsilon xW' \cap W'x$ and since $xW' \cap W'x \subset xW'$ and $xW' \cap W'x \subset W'x$, we have that $xW' = xW' \cap W'x = W'x$. Therefore, W is an invariant subgroup of G.

Let $H = W'$. Let $T \varepsilon \mathcal{J} \ni T \neq \emptyset$ and let $x' \varepsilon T$. Since $H \subset G$ and $e \varepsilon H$, $G = \bigcup_{x \varepsilon G} xH$. Then $T \cap x'H \neq \emptyset$, $T \cap x'H \subset x'H$, and $T \cap x'H \varepsilon \mathcal{J}$. Hence, by definition of $\mathcal{W}$, $T \cap x'H = x'H$, which $\Rightarrow$ $T \supset x'H$. Hence, $T = \bigcup_{x \varepsilon T} xH$. Therefore, $T \varepsilon \mathcal{J} \Leftrightarrow T = \emptyset$ or $T = \bigcup_{x \varepsilon T} xH$.

Therefore, $\mathcal{J}$ is the topology generated by the invariant subgroup H of G. ||

COROLLARY 3. IF $(G, \cdot, \mathcal{J})$ is a topological group, then $\mathcal{J}$ is the topology generated by the invariant subgroup H of $G \Leftrightarrow H \varepsilon \mathcal{J}$ and no nonempty member of $\mathcal{J}$ is a proper subset of H. Furthermore, if $\mathcal{J}$ is generated by H and if $W' \varepsilon \mathcal{J} \ni W' \neq \emptyset$, no nonempty member of $\mathcal{J}$ is a proper subset of W, and $e \varepsilon W'$, then $W' = H$.

<u>Proof</u>: This follows at once from the definition of a topology generated by an invariant subgroup and the proof of the above theorem. ||

COROLLARY 4. Let $(G,\cdot,\mathcal{J})$ be a topological group. If $\mathcal{J}$ is discrete or if $\mathcal{J}$ is finite, then $\mathcal{J}$ is the topology generated by an invariant subgroup H of G.

<u>Proof</u>: If $\mathcal{J}$ is discrete, then the subgroup of G consisting of the neutral element e of G is a member of $\mathcal{J}$ and properly contains no nonempty member of $\mathcal{J}$. Hence, by Corollary 3, above, $\mathcal{J}$ is the topology generated by {e}.

Suppose $\mathcal{J}$ is finite, containing, say, precisely n sets. Suppose $\exists\ T \in \mathcal{J} \ni T \neq \emptyset$ and $\ni \forall\ W \in \mathcal{J} \ni W \neq \emptyset$ and $W \subset T$, $\exists$ a nonempty member W' of $\mathcal{J} \ni$ W' is a proper subset of W.

Since $\mathcal{J}$ is finite, $\exists\ m \in N \ni m \leq n$ and $\ni \exists$ only m distinct open subsets of T. But, by our hypothesis, each of the subsets contains as a proper subset some member of T. We conclude that $\mathcal{J}$ finite $\Rightarrow \forall\ T \in \mathcal{J} \ni T \neq \emptyset$, $\exists\ W \in \mathcal{J} \ni W \neq \emptyset$, $W \subset T$, and no nonempty member of $\mathcal{J}$ is a proper subset of W.

Therefore, $\mathcal{J}$ is the topology generated by an invariant subgroup H of G. ||

Theorem 5. Let $(G,\cdot,\mathcal{J})$ be a topological group $\ni \mathcal{J}$ is the topology generated by the invariant subgroup H of G. If G/H is finite, containing, say, precisely n distinct elements, then $\exists$ precisely 2^n distinct elements in $\mathcal{J}$.

<u>Proof</u>: Suppose G/H is finite. Then $\exists$ a 1-1 correspondence between members of $\mathcal{J}$ and subsets of G/H. Hence, if $\exists$ precisely n distinct elements in G/H, $\exists$ precisely 2^n distinct sets in $\mathcal{J}$. ||

LEMMA 6. Let $(G,\cdot,\mathcal{J})$ be a topological group $\ni \{e\} \in \mathcal{J}$, where e is the neutral element of G. Then $\mathcal{J}$ is the discrete topology for G.

<u>Proof</u>: By Theorem I.1, $\forall\, a \in G$, L_a is a homeomorphism. Hence, if $\{e\} \in \mathcal{J}$, then $\forall\, a \in G$, $L_a[\{e\}] = \{a\} \in \mathcal{J}$.

Therefore, $\{e\} \in \mathcal{J} \Rightarrow \mathcal{J}$ discrete. ||

THEOREM 7. Let $(G,\cdot,\mathcal{J},\Phi)$ be a topological group with operators $\ni \mathcal{J}$ is the topology generated by an invariant subgroup H of G. If $\alpha[H] \subset H$, $\forall\, \alpha \in \Phi$(as will be the case if $\Phi = \emptyset$, for example), then all quotient groups of Jordan-Hölder series for G are either discrete or indiscrete with the quotient topology. Specifically, if $G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$, where e is the neutral element of G, is a Jordan-Hölder series for G, then, for $0 \leq i \leq n-1$, the quotient topology $\mathcal{J}_{G_i/G_{i+1}}$ for G_i/G_{i+1} is discrete $\Leftrightarrow H \cap G_i \subset G_{i+1}$.

<u>Proof</u>: Let $(G,\cdot,\mathcal{J},\Phi)$ be a topological group $\ni \mathcal{J}$ is the topology generated by H, where H is an invariant subgroup of G $\ni$ $\alpha[H] \subset H$, $\forall\, \alpha \in \Phi$. Let $G = G_0 \supset G_1 \supset \dots \supset G_n = \{e\}$ be any Jordan-Hölder series for G and let G_i/G_{i+1} be one of the quotient groups of this series.

We first wish to show that any subset A of $G_i/G_{i+1} \ni \mathcal{A} = \bigcup_{\bar{a} \in A} \bar{a} \supset H \cap G_i$ and $\ni \forall\, \bar{a} \in A$, $\bar{a} \cap (H \cap G_i) \neq \emptyset$ is an invariant subgroup of $G_i/G_{i+1} \ni \alpha[A] \subset A$, $\forall\, \alpha \in \Phi$.

Clearly, such a subset A is not empty, since $e \in H \cap G_i \Rightarrow \bar{e} \in A$. Let $\bar{a}_1, \bar{a}_2 \in A$. Then, by definition of A, $\exists\ a_1 \in \bar{a}_1, a_2 \in \bar{a}_2 \ni a_1, a_2 \in H \cap G_i$. Since H and G_i are subgroups of G, $a_1, a_2 \in H \cap G_i \Rightarrow a_1, a_2^{-1} \in H \cap G_i$, which $\Rightarrow a_1 a_2^{-1} \in H \cap G_i$. Hence, by Theorem I.11, H is a subgroup of G_i/G_{i+1}.

Since H is an invariant subgroup of G and G_i is an invariant subgroup of itself, we have that, $\forall\ g \in G_i$, $g(H \cap G_i) = gH \cap gG_i = Hg \cap G_i g = (H \cap G_i)g$. Thus, $H \cap G_i$ is invariant.

Let $\bar{g} \in G_i/G_{i+1}$. Then $\bar{g} \subset G_i$. Let $\bar{x} \in \bar{g}A$. Then $\exists\ \bar{a} \in A \ni \bar{x} = \bar{g}\bar{a}$. But, $\bar{a} \in A \Rightarrow \exists\ a \in \bar{a} \ni a \in H \cap G_i$. Let $g \in \bar{g}$. Then $ga \in g(H \cap G_i) = (H \cap G_i)g$, which $\Rightarrow \exists\ a' \in H \cap G_i \ni ga = a'g$. And, $H \cap G_i \subset A \Rightarrow a' \in A$. Hence, since unequal cosets are distinct, $\bar{x} = \bar{g}\bar{a} = \bar{a}'\bar{g} \in Ag$, which $\Rightarrow \bar{g}A \supset A\bar{g}$. The proof that $\bar{g}A \subset A\bar{g}$ is, clearly, entirely analogous to the foregoing. Therefore, A is an invariant subgroup of G.

By an assumption concerning H and by definition of Jordan-Hölder series, $\alpha[H] \subset H$ and $\alpha[G_i] \subset G_i$, $\forall\ \alpha \in \Phi$. Let $x \in \alpha[H \cap G_i]$. Then $\exists\ h \in H$ and $g \in G_i \ni x = \alpha(h) = \alpha(g)$, which $\Rightarrow x \in H \cap G_i$. Hence, $\forall\ \alpha \in \Phi$, $\alpha[H \cap G_i] \subset \alpha[H]$. Let $\alpha \in \Phi$ and $\bar{a} \in A \ni a \in H \cap G_i$. Then $\alpha(\bar{a}) = \alpha(aG_{i+1}) = \alpha(a)G_{i+1} \subset G_{i+1} \subset G_i$. Since H is invariant and $a \in H$, $aG_{i+1} \subset H$, which $\Rightarrow \alpha(\bar{a}) \subset H$. So, $\alpha(\bar{a}) \subset H \cap G_i$, $\forall\ a \in A$.

Therefore, $\forall\ \alpha \in \Phi$, $\alpha[A] \subset A$.

But, G_i/G_{i+1} is simple (see Theorem I.10). Hence, $A = G_i/G_{i+1}$ or $A = \{e\}$. Since $H \cap G_i \subset G_i$, it is clear that such a set A exists. Suppose $A = \{e\}$, then, by definition of A, $\bar{e} = G_{i+1} \supset H \cap G_i$. Therefore, there exists only one such set A.

Suppose $A = \{e\}$. Then $\mathcal{Q} = \bar{e} \supset H \cap G_i \in \mathcal{J}_{G_i}$. Let $x \in \bar{e}$. Then $x \in G_i$ and $x \in xH \cap G_i$. Let $y \in xH \cap G_i$. Then $\exists\ h \in H \cap G_i \ni y = xh$. And, $h \in H \cap G_i \Rightarrow h \in \bar{e}$, which $\Rightarrow y = xh \in \bar{e}$ since $\bar{e} = G_{i+1}$ is a group. Hence, $xH \cap G_i \subset \bar{e}$. Thus, $\mathcal{Q} = \bar{e} = \bigcup_{x \in \bar{e}} (xH \cap G_i) = (\bigcup_{x \in \bar{e}} xH) \cap G_i \in \mathcal{J}_{G_i}$. Therefore, $\{\bar{e}\} \in \mathcal{J}_{G_i/G_{i+1}}$ and we have by the above lemma that $\mathcal{J}_{G_i/G_{i+1}}$ is discrete.

Suppose $A = G_i/G_{i+1}$. Let $B \in \mathcal{J}_{G_i/G_{i+1}} \ni B \neq \emptyset$. Then the set $\mathcal{B} = \bigcup_{\bar{b} \in B} \bar{b} \in \mathcal{J}_{G_i}$ and $\exists$ a subset D of G $\ni \mathcal{B} = (\bigcup_{d \in D} dH) \cap G_i$. Let $d' \in D$. Let $\bar{x} \in B \ni \exists\ b \in \bar{x} \ni b \in d'H \cap G_i$. Let $\bar{y} \in G_i/G_{i+1}$. Then $\exists\ \bar{z} \in G_i/G_{i+1} \ni \bar{y}\bar{z} = \bar{x}$ (since G_i/G_{i+1} is a group). Let $z \in \bar{z} \ni z \in H \cap G_i$ (such a z exists since $A = G_i/G_{i+1}$). Then $\exists\ y \in \bar{y} \ni yz = b$, which $\Rightarrow y = bz^{-1}$. Since $z \in H \cap G_i \Rightarrow z^{-1} \in H \cap G_i$ (since $H \cap G_i$ is a group), we have that $y = bz^{-1} \in (dH \cap G_i)(H \cap G_i) = dH \cap G_i \subset \mathcal{B}$. Hence, $\bar{y} \cap \mathcal{B} \neq \emptyset$, which $\Rightarrow \bar{y} \in B$. Thus, $B = G_i/G_{i+1}$. Therefore, $\mathcal{J}_{G_i/G_{i+1}}$

is indiscrete.

This completes the proof of the theorem.

By the Jordan-Hölder Theorem, if $(G,\cdot,\Phi)$ and $(G',\cdot,\Phi)$ are groups with operators $\ni$ $\exists$ non-equivalent Jordan-Hölder series Σ and Σ' for G and G', respectively, then G and G' are not isomorphic.

Let $(G,\cdot,\mathcal{J},\Phi)$ and $(G',\cdot,\mathcal{J}',\Phi)$ be topological groups with operators $\ni$ G and G' are isomorphic. Then, by Theorem II.3, if $\exists$ Jordan-Hölder series Σ and Σ' for G and G', respectively, $\ni$ Σ and Σ' are not topologically equivalent, $(G,\mathcal{J})$ and $(G,\mathcal{J}')$ are not homeomorphic topological spaces. A partial converse also holds, as follows:

THEOREM 8. Let $(G,\cdot,\mathcal{J},\Phi)$ and $(G',\cdot,\mathcal{J}',\Phi)$ be topological groups with operators $\ni$ $\mathcal{J}$ and $\mathcal{J}'$ are the topologies generated by the invariant subgroups H and J' of G and G', respectively, $\ni$ G and G' are isomorphic. Suppose, further, that $\forall\ \alpha \in \Phi$, $\alpha[H] \subset H$ and $\alpha[J'] \subset J'$. Then, if $\exists$ Jordan-Hölder series Σ and Σ' for G and G', respectively, $\ni$ Σ and Σ' are topologically equivalent, $(G,\mathcal{J})$ and $(G,\mathcal{J}')$ are homeomorphic topological spaces. Specifically, two finite groups with the null set of operators which are isomorphic are abstractly identical $\Leftrightarrow$ a Jordan-Hölder series for one is topologically equivalent to a Jordan-Hölder series for the other.

<u>Proof</u>: Let $(G,\cdot,\mathcal{J},\Phi)$ and $(G',\cdot,\mathcal{J}',\Phi)$ be the topological groups described in the theorem. Clearly, if $\exists$ topologically equivalent Jordan-Hölder series Σ and Σ' for G and G', respectively, then any Jordan-Hölder series for G is topologically equivalent to

any Jordan-Hölder series for G'.

Let f be an isomorphism of the group G onto the group G' and, $\forall$ a ε G, denote f(a) by a'. Then, changing the notation of G', if necessary, we can write a' = a, $\forall$ a ε G.

Since H is an invariant subgroup of G $\ni \alpha[H] \subset H$, $\forall \alpha \varepsilon \Phi$, $\exists$ a Jordan-Hölder series $\Sigma = (G_i)_{0\leq i\leq n}$ for G $\ni$ H = G_k for some k ε {0,1,...,n}. Then $\Sigma' = (f[G_i])_{0\leq i\leq n} = (G_i')_{0\leq i\leq n}$ is a Jordan-Hölder series for G' $\ni G_i = G_i'$, $\forall$ i ε {0,1,...,n} and, specifically, H = H' = G_k'.

Since Σ and Σ' are topologically equivalent, by hypothesis, we have (see Lemma II.2 and Theorem II.3), $\forall$ j ε {0,1,...,n}, the quotient space $G_{j,j} = (G_j \cap G_j')G_{j+1}/(G_j \cap G_j')G_{j+1} = G_jG_{j+1}/G_{j+1}G_{j+1} = G_j/G_{j+1}$ is isomorphic and homeomorphic to the quotient space $G_{j,j}' = (G_j \cap G_j')G_{j+1}'/(G_{j+1} \cap G_j')G_{j+1}' = G_j'G_{j+1}'/G_{j+1}'G_{j+1}' = G_j'/G_{j+1}'$.

$\forall$ j ε {1,...,k}, G_i is a subgroup of G $\ni$ H $\subset G_i$. Let j ε {1,...,k} and let $\bar{e}$ be the neutral element of G_{j-1}/G_j. Then $\bar{e}$ = $G_j = \bigcup_{x \varepsilon G_j} xH$, which => $\bigcup_{a \varepsilon \bar{e}}\{a\}$ is open in (G,$\mathcal{J}$), which => $\bar{e}$ is open in G_{j-1}/G_j, which => (see Lemma 6) G_{j-1}/G_j is discrete, which => $\forall$ i ε {1,...,k}, G_{i-1}'/G_i' is discrete.

Specifically, then, G'/G_1' is discrete, which => $G_1' \varepsilon \mathcal{J}'$. And, G_1'/G_2' discrete => G_2' open in G_1', which => $\exists$ U' $\varepsilon \mathcal{J}' \ni U' \cap G_2' = G_1'$, which => $G_1' \varepsilon \mathcal{J}'$. Similarly, it can be shown that $G_2', G_3', \ldots, G_k' \varepsilon \mathcal{J}'$.

Hence, $G'_k = H' \in \mathcal{J}'$, which => $H = H' \supset J'$.

Since f is an isomorphism of G onto G', f^{-1} is an isomorphism of G' onto G. Hence, we can show in an analogous manner that $J = f^{-1}[J'] = J' \supset H = H'$.

Therefore, $J' = H'$ and we have that $\mathcal{J}$ and $\mathcal{J}'$ are homeomorphic. Specifically, if G and G' are finite and $\Phi = \emptyset$, we have by Corollary 4 that the above result holds. ||

IV. MAPPINGS OF A TOPOLOGICAL GROUP ONTO ITSELF

DEFINITION. We define an <u>auteomorphism</u> as a homeomorphism of a topological space onto itself.

DEFINITION. Let $(G,\cdot,\mathcal{J})$ be a topological group. We define an <u>autoauteomorphism</u> of $(G,\cdot,\mathcal{J})$ onto itself as an auteomorphism of $(G,\mathcal{J})$ which is also an automorphism of $(G,\cdot)$, i.e., an auteomorphism α of $(G,\mathcal{J}) \ni \forall\, x, y \in G$, $\alpha(xy) = \alpha(x)\alpha(y)$.

THEOREM 1. The set $\mathcal{Q}$ of all auteomorphisms of a topological group $(G,\cdot,\mathcal{J})$ forms a group $(\mathcal{Q},\cdot)$ under composition of functions. The set $\mathcal{Q}'$ of all autoauteomorphisms of $(G,\cdot,\mathcal{J})$ forms a subgroup $(\mathcal{Q}',\cdot)$ of this group.

<u>Proof</u>: If $\alpha,\beta \in \mathcal{Q}$, then (see Theorem I.13) $\alpha\beta,\alpha^{-1} \in \mathcal{Q}$. And, the identity mapping ι of G onto itself is an auteomorphism of G. Hence, (see Theorem I.11) $(\mathcal{Q},\cdot)$ is a subgroup of the group of 1-1 mappings of G onto itself and, therefore, a group.

By definition, $\mathcal{Q}' \subset \mathcal{Q}$. Hence, if $\alpha,\beta \in \mathcal{Q}'$, then $\alpha\beta, \alpha^{-1} \in \mathcal{Q}$. Since (see Theorem I.13) the composite of two isomorphisms is an isomorphism and the inverse of an isomorphism is an isomorphism, we have, then, that $\alpha,\beta \in \mathcal{Q}' \Rightarrow \alpha\beta, \alpha^{-1} \in \mathcal{Q}'$. Clearly, $\iota \in \mathcal{Q}'$. Hence, (see Theorem I.11) $(\mathcal{Q}',\cdot)$ is a subgroup of the group of 1-1 mappings of G onto G and, therefore, a subgroup of $(\mathcal{Q},\cdot)$. ||

THEOREM 2. Let $(G,\cdot,\mathcal{J})$ be a topological group. Then the set $\mathcal{I}$ of all inner automorphisms of $(G,\cdot)$ forms an invariant subgroup $(\mathcal{I},\cdot)$ the group $(\mathcal{Q}',\cdot)$ of autoauteomorphisms of $(G,\cdot)$.

<u>Proof</u>: Let $\alpha \in \mathcal{I}$. Then, by definition of inner automorphism, $\exists\ a \in G \ni, \forall\ x \in G,\ \alpha(x) = a\cdot x\cdot a^{-1}$. But, $a\cdot x\cdot a^{-1} = L_a \cdot R_{a^{-1}}(x)$, where L_a and $R_{a^{-1}}$ are, respectively, the left translation by a and the right translation by a^{-1}. Hence, α is a composite of homeomorphisms (Theorem I.1) and, therefore, a homeomorphism. Therefore, since inner automorphisms are automorphisms (Theorem I.14), $\mathcal{I} \subset \mathcal{Q}'$.

And, $\mathcal{Q}'$ is, by definition, a subset of the set of all automorphisms of G. Finally (see Theorem I.14), $(\mathcal{I},\cdot)$ is an invariant subgroup of the set of all automorphisms of G. Hence, $(\mathcal{I},\cdot)$ is an invariant subgroup of $(\mathcal{Q}',\cdot)$. ||

THEOREM 3. Let $(G,\cdot,\mathcal{J})$ be any topological group and let $(\mathcal{Q},\cdot)$ be the group of auteomorphisms of $(G,\mathcal{J})$. $\forall\ U_1, U_2 \in \mathcal{J}$, define the set $B_{U_1,U_2} = \{\alpha \in \mathcal{Q} \mid \alpha[U_1] = U_2\}$. Let 2N denote the set of all even positive integers. Let $\mathcal{B}$ be the family of subsets of $\mathcal{Q}$ defined as follows: $B \in \mathcal{B} \Leftrightarrow \exists$ a finite sequence $U_1, U_2,\ldots,U_n$, $n \in 2N$, of members of $\mathcal{J} \ni B = B_{U_1,U_2} \cap B_{U_3,U_4} \cap \ldots \cap B_{U_{n-1},U_n}$.
Then $\mathcal{B}$ is a base for a topology $\mathcal{J}_{\mathcal{B}}$ for $\mathcal{Q} \ni (\mathcal{Q},\cdot,\mathcal{J}_{\mathcal{B}})$ is a topological group.

<u>Proof</u>: $\forall\ U_1, U_2 \in \mathcal{J}$, $B_{U_1,U_2} \subset \mathcal{Q}$ and $\alpha \in \mathcal{Q} \Rightarrow B_{U,\alpha[U]} \in \mathcal{B}$, $\forall\ U \in \mathcal{J}$. Hence, $\mathcal{Q} = \bigcup_{U_1,U_2 \in \mathcal{J}} B_{U_1,U_2}$.

Clearly, the intersection of two members of $\mathcal{B}$ is a member of $\mathcal{B}$. Hence, (see Theorem I.15) $\mathcal{B}$ is a base for a topology $\mathcal{J}_{\mathcal{B}}$ for $\mathcal{Q}$.

Let $\alpha_1, \alpha_2 \in \mathcal{Q}$. Let W be any nbd of $\alpha_1 \cdot \alpha_2^{-1}$. Then, by definition of a base for a topology (see p.2), $\exists\ B \in \mathcal{B} \ni \alpha_1 \cdot \alpha_2^{-1} \in B \subset W$. Then $\exists\ n \in 2N$ and $U_1, U_2, \ldots, U_m \in \mathcal{J} \ni B = B_{U_1,U_2} \cap B_{U_3,U_4} \cap \ldots \cap B_{U_{m-1},U_m}$.

Let $i \in 2N \ni i \leq m$. Then $\alpha_1[\alpha_2^{-1}[U_{i-1}]] = U_i$ and $\alpha_2[\alpha_2^{-1}[U_{i-1}]] = U_{i-1}$. Hence, $B_1 = B_{\alpha_2^{-1}[U_1],U_2} \cap B_{\alpha_2^{-1}[U_3],U_4} \cap \ldots \cap B_{\alpha_2^{-1}[U_{m-1}],U_m}$ is a nbd of α_1 and $B_2 = B_{\alpha_2^{-1}[U_1],U_1} \cap B_{\alpha_2^{-1}[U_3],U_3} \cap \ldots \cap B_{\alpha_2^{-1}[U_{m-1}],U_{m-1}}$ is a nbd of α_2.

$$\begin{aligned} B_2^{-1} &= \{\alpha \in \mathcal{Q} \mid \alpha^{-1}[\alpha_2^{-1}[U_1]] = U_1, \alpha^{-1}[\alpha_2^{-1}[U_3]] = U_3, \ldots, \alpha^{-1}[\alpha_2^{-1}[U_{m-1}]] = U_{m-1}\} \\ &= \{\alpha \in \mathcal{Q} \mid \alpha[U_1] = \alpha_2^{-1}[U_1], \alpha[U_3] = \alpha_2^{-1}[U_3], \ldots, \alpha[U_{m-1}] = \alpha_2^{-1}[U_{m-1}]\} \\ &= B_{U_1,\alpha_2^{-1}[U_1]} \cap B_{U_3,\alpha_2^{-1}[U_3]} \cap \ldots \cap B_{U_{n-1},\alpha_2^{-1}[U_{m-1}]}. \end{aligned}$$

Let $\theta \in B_1 \cdot B_2^{-1}$. Then $\exists\ \beta \in B_1$, $\gamma \in B_2^{-1} \ni \theta = \beta\gamma$. Let $i \in 2N \ni i \leq m$. Then $\gamma[U_{i-1}] = \alpha_2^{-1}[U_{i-1}], \beta[\alpha_2^{-1}[U_{i-1}]] = U_i$; which

$\Rightarrow \vartheta[U_{i-1}] = \beta[\vartheta[U_{i-1}]] = U_i'$, which $\Rightarrow \vartheta \in B$. Hence, $B_1 \cdot B_2^{-1} \subset B$.

Therefore, $(\mathcal{Q},\cdot,\mathcal{J}_{\mathcal{B}})$ is a topological group. ||

COROLLARY 4. Let $\mathcal{U}_{\mathcal{B}}$ be the nbd system of ι in $(\mathcal{Q},\cdot,\mathcal{J}_{\mathcal{B}})$. Then $U \in \mathcal{U}_{\mathcal{B}} \Leftrightarrow \exists B \in \mathcal{B} \ni B = B_{U_1,U_1} \cap B_{U_2,U_2} \cap \ldots \cap B_{U_n,U_n}$, where $U_1,\ldots,U_n \in \mathcal{J}$, $\ni B \subset U$.

<u>Proof</u>: Let $U \in \mathcal{U}_{\mathcal{B}}$. Then $\iota \in U$, which $\Rightarrow \exists B \in \mathcal{B} \ni \iota \in B \subset U$. And, $B \in \mathcal{B} \Rightarrow \exists m \in 2N \ni B = B_{U_1,U_2} \cap B_{U_3,U_4} \cap \ldots \cap B_{U_{m-1},U_m}$ for $U_1,U_2,\ldots,U_m \in \mathcal{J}$. Suppose $\exists i \in 2N \ni i \leq m$ and $\ni U_{i-1} \neq U$. Then $\iota[U_{i-1}] \neq \iota[U_i]$, which $\Rightarrow \iota \notin B$, contrary to hypothesis.

Suppose $U \subset \mathcal{Q} \ni \exists B \in \mathcal{B}$ with the given properties. Then $\iota \in B \subset U$, which $\Rightarrow U \in \mathcal{U}_{\mathcal{B}}$. ||

THEOREM 5. Let $(G,\cdot,\mathcal{J})$ be a topological group $\ni \mathcal{J}$ is the topology generated by the invariant subgroup H of G. Let $(\mathcal{Q},\cdot,\mathcal{J}_{\mathcal{B}})$ be the topological group of auteomorphisms of $(G,\mathcal{J})$ described in Theorem 3 above. Then the following are equivalent:

(1) $\mathcal{J}$ is finite.

(2) $\mathcal{J}_{\mathcal{B}}$ is the topology generated by an invariant subgroup $(A,\cdot)$ of $(\mathcal{Q},\cdot)$.

(3) $(\mathcal{Q},\mathcal{J}_{\mathcal{B}})$ is compact.

Specifically, if $\mathcal{J}_{\mathcal{B}}$ is generated by an invariant subgroup $(A,\cdot)$ of $(\mathcal{Q},\cdot)$, then $A = \bigcap_{aH \in G/H} B_{aH,aH}$.

<u>Proof</u>: <u>(1) => (2)</u>. This is an immediate consequence of Corollary III.4.

<u>(2) => (1)</u>. Suppoes that $\mathcal{J}$ is not finite. Let $T \in \mathcal{J}_{\mathcal{B}} \ni T \neq \emptyset$. Let $W \in \mathcal{J}_{\mathcal{B}} \ni W \subset T$ and $W \neq \emptyset$. Let $\alpha \in W$. Then $\exists\ U_1, U_2, \ldots, U_n \in \mathcal{J}$, for $n \in 2N$, $\ni \alpha \in B = B_{U_1,U_2} \cap B_{U_3,U_4} \cap \ldots \cap B_{U_{n-1},U_n} \subset W$. Since $\mathcal{J}$ is infinite, $\exists\ U_j, U_k \in \mathcal{J} \ni U_j \notin \{U_1, U_3, \ldots, U_{n-1}\}$, $U_k \notin \{U_2, U_4, \ldots, U_n\}$, and $B_{U_j,U_k} \neq \emptyset$. Then $B' = B \cup B_{U_j,U_k} \neq \emptyset$. Hence, $B' \in \mathcal{J}_{\mathcal{B}} \ni B' \neq \emptyset$, $B' \subset W$, and $B' \neq W$. Therefore, by Theorem III.2, $\mathcal{J}_{\mathcal{B}}$ is not the topology generated by an invariant subgroup of $(\mathcal{Q}, \cdot)$.

<u>(1) => (3)</u>. Suppose $\mathcal{J}$ is finite. Let $U = (U_i)_{i \in I}$, for some index set I, be any convering of $\mathcal{Q}$ by nonempty members of $\mathcal{J}_{\mathcal{B}}$. Then, $\forall\ i \in I$, $\exists$ an index set J_i and a family $B_i = (B_{i_j})_{j \in J_i}$ of members of $\mathcal{B} \ni U_i = {}_{j \in J_i} B_{i_j}$. Let $B = (B_{i_j})_{\substack{i \in I \\ j \in J}}$. Then B is a covering of $\mathcal{Q}$.

Suppose $\exists$ precisely n distinct sets in $\mathcal{J}$. Then $\exists$ precisely n-1 distinct nonempty sets in $\mathcal{J}$ and $\exists\ m \in N \ni m \leq (n-1)^2 \ni \exists$ precisely m distinct members of $\mathcal{B}$ of the form B_{U_1,U_2}, for $U_1, U_2 \in \mathcal{J}$. And, in a manner entirely analogous to that used in Theorem III.5, it can be shown that $\exists$ precisely 2^m-1 distinct sets in $\mathcal{B}$, which => $\mathcal{B}$ is finite. Hence, by omitting, if necessary, any repetitions of sets in B, we can obtain a finite subcovering $B' = (B_k)_{k \in K}$ of B, where $K \subset \{i_j \mid i \in I, j \in J_i\}$. Then the family U' =

$\{U_i | i \in I \ni \exists\, j \in J_i \ni i_j \in K\}$ is a finite subfamily of U which covers $\mathcal{Q}$. Therefore, $\mathcal{J}_{\mathcal{B}}$ is compact.

<u>(3) => (1).</u> Suppose $\mathcal{J}_{\mathcal{B}}$ is compact. Let $B = \{B_{H,U} | U \in \mathcal{J}\}$. Since $H \in \mathcal{J}$, $B_{H,U} \in \mathcal{B} \subset \mathcal{J}_{\mathcal{B}}$, $\forall\, U \in \mathcal{J}$, and $\alpha[H] \in \mathcal{J}$, $\forall\, \alpha \in \mathcal{Q}$. Hence, B is an open covering of $\mathcal{Q}$. Thus, since $\mathcal{J}_{\mathcal{B}}$ is compact, $\exists$ a finite subfamily B' of B which covers $\mathcal{Q}$. Then $\exists\, n \in N \ni B'$ contains precisely n members. Let $a \in G$. Let α be the mapping of G onto G $\ni \alpha(x) = x$, $\forall\, x \in G \ni x \notin H$ and $x \notin aH$, $\alpha(x) = ax$, $\forall\, x \in H$, and $\alpha(x) = a^{-1}x$, $\forall\, x \in aH$. Then, clearly, $\alpha \in A \ni \alpha[H] = aH$, which => $\alpha \in B_{H,U}$ <=> $U = aH$. Therefore, $B_{H,aH} \in B'$, $\forall\, a \in G$. Hence, $\exists\, m \in N \ni m \leq n$ and $\ni \exists$ precisely m distinct members of G/H, which => (by Theorem III.5) precisely 2^m distinct members of $\mathcal{J}$. Therefore, $\mathcal{J}$ is finite.

Finally, suppose that $\mathcal{J}_{\mathcal{B}}$ is generated by an invariant subgroup of $(\mathcal{Q}, \cdot)$. Let $A = \bigcap_{aH \in G/H} B_{aH,aH}$. Clearly, $A \neq \emptyset$ and $A \in \mathcal{B} \subset \mathcal{J}_{\mathcal{B}}$. Suppose $\exists\, T \in \mathcal{J} \ni T \neq \emptyset$ and T is a proper subset of A. $T \in \mathcal{J}$ => $\exists$ $U_1, U_2, \ldots, U_n \in \mathcal{J}$, for $n \in 2N$, $\ni B = B_{U_1,U_2} \cap B_{U_3,U_4} \cap \ldots \cap B_{U_{n-1},U_n} \subset T$.

Let $\alpha \in A$. Let $i \in 2N \ni i \leq n$. If $U_{i-1} = \emptyset$, then $B \subset \mathcal{Q} \Rightarrow \beta[U_{i-1}] = U_{i-1}$, $\forall\, \beta \in B$, which => $U_i = U_{i-1}$. If $U_{i-1} \neq \emptyset$, then $a_1H, a_2H, \ldots, a_mH \in G/H \ni U_{i-1} = a_1H \cup a_2H \cup \ldots \cup a_mH$. And, $B \subset A$ =>, $\forall\, \beta \in B$, $\beta[U_{i-1}] = \beta[a_1H \cup \ldots \cup a_mH] = \beta[a_1H] \cup \ldots \cup \beta[a_mH] = a_1H \cup \ldots \cup a_mH = U_{i-1}$. Hence, $\forall\, i \in 2N \ni i \leq n$, $U_i = U_{i-1}$. Since

$\alpha \in A$, $\alpha[U_{i-1}] = U_{i-1}$, $\forall\, i \in \{0,1,\ldots,n\}$, which => $\alpha \in B$. Hence, $B = A$, which => $T = A$, contrary to hypothesis.

Therefore, by Corollary III.3, $(A,\cdot)$ is the invariant subgroup of $(\mathcal{Q},\cdot)$ which generates $\mathcal{J}_{\mathcal{B}}$. ||

THEOREM 6. Let $(G,\cdot,\mathcal{J})$ be a topological group and let $(\mathcal{Q},\cdot,\mathcal{J}_{\mathcal{B}})$ be the topological group of auteomorphisms of $(G,\mathcal{J})$ described in Theorem 3, above. Then the following hold:

(1) $\mathcal{J}_{\mathcal{B}}$ is discrete <=> G is finite and $\mathcal{J}$ is discrete,

(2) $\mathcal{J}_{\mathcal{B}}$ is indiscrete <=> $\mathcal{J}$ is indiscrete,

(3) $(\mathcal{Q},\mathcal{J}_{\mathcal{B}})$ is a T_i-space $(0 \leq i \leq 3)$ <=> $(G,\mathcal{J})$ is a T_2-space.

<u>Proof</u>: Suppose G is finite, say $G = \{a_1,\ldots,a_n\}$, and $\mathcal{J}$ is discrete. Then $\{\iota\} = B_{\{a_1\},\{a_1\}} \cap \ldots \cap B_{\{a_n\},\{a_n\}} \in \mathcal{J}_{\mathcal{B}}$. Hence, by Lemma III.6, $\mathcal{J}_{\mathcal{B}}$ is discrete. Conversely, suppose G is finite, but $\mathcal{J}$ is not discrete. Then $\exists\, U \in \mathcal{J} \ni U$ contains more than one element, say $U = \{a_1,\ldots,a_n\}$, and since G is finite, we can choose $U \ni U$ properly contains no nonempty member of $\mathcal{J}$. Let $B = B_{U_1,U_1} \cap \ldots \cap B_{U_n,U_n} \in \mathcal{B} \ni \iota \in B$ and $\ni \exists\, k \in \{1,\ldots n\} \ni U = U_k$. Define the mapping $\beta \ni \beta(x) = \iota(x)$, $\forall\, x \in G \ni x \notin U_k$, $\beta(a_i) = a_{i+1}$ for $a_i \in U$, $1 \leq i \leq n$, and $\beta(a_n) = a_1$. Clearly, $\beta \in B$. Hence, $\{\iota\} \notin \mathcal{J}_{\mathcal{B}}$, which => $\mathcal{J}_{\mathcal{B}}$ is not discrete. Therefore, (1) holds.

Suppose $\mathcal{J}$ is indiscrete. Then $\mathcal{B} = \{B_{G,G}, B_{\emptyset,\emptyset}\} = \{\mathcal{Q},\emptyset\}$, which $\Rightarrow$ $\mathcal{J}_{\mathcal{B}}$ is indiscrete. Conversely, if $\mathcal{J}$ is not indiscrete, then $\exists\ U \in \mathcal{J} \ni U \neq \emptyset$ and $U \neq G$. Hence, $\forall\ u \in U$, $u^{-1}U$ is a nbd of the neutral element e of G which is neither empty nor G. (Note that if $u^{-1}U = G$, then $G = uG = U$.) Let $x \in G \ni x \notin u^{-1}U$. Then $L_x \in B_{u^{-1}U,xu^{-1}U}$, which $\Rightarrow B_{u^{-1}U,xu^{-1}U} \neq \emptyset$. Since ι is not a member of this set, it is a member of $\mathcal{B}$ and, hence, of $\mathcal{J}_{\mathcal{B}}$ which is not $\emptyset$ and not $\mathcal{Q}$. Thus, $\mathcal{J}_{\mathcal{B}}$ is not indiscrete. Therefore, (2) holds.

By Theorem I.5, a topological group is a T_2-space whenever it is a T_0-space. Suppose $(G,\mathcal{J})$ is a T_2-space. Let $\alpha,\beta \in \mathcal{Q} \ni \alpha \neq \beta$. Then $\exists\ x \in G \ni \alpha(x) \neq \beta(x)$, which $\Rightarrow \exists$ nbds $U_{\alpha(x)}$ and $U_{\beta(x)}$ of $\alpha(x)$ and $\beta(x)$, respectively, $\ni U_{\alpha(x)} \cap U_{\beta(x)} = \emptyset$. And, for convenience, we can assume that these nbds are open. Since α and β are auteomorphisms, $\alpha^{-1}U_{\alpha(x)}$ and $\beta^{-1}U_{\beta(x)}$ are open nbds of x, which $\Rightarrow$ the set $A = \alpha^{-1}U_{(x)} \cap \beta^{-1}U_{(x)}$ is an open nbd of x. Hence, $\alpha[A]$ and $\beta[A]$ are open. And, $\alpha[A] \subset U_{\alpha(x)}$, $\beta[A] \subset U_{\beta(x)} \Rightarrow \alpha[A] \cap \beta[A] = \emptyset$, which $\Rightarrow B_{A,\alpha[A]} \cap B_{A,\beta[A]} = \emptyset$. Clearly, $B_{A,\alpha[A]}$ and $B_{A,\beta[A]}$ are open nbds of α and β, respectively. Hence, $(\mathcal{Q},\mathcal{J}_{\mathcal{B}})$ is a T_2-space.

Conversely, suppose $(G,\mathcal{J})$ is not a T_2-space. Then it is not a T_0-space. Hence, $\exists$ x, $y \in G \ni x \neq y$ and $\ni$ every nbd of x is a nbd of y and every nbd of y is a nbd of x.

Define the mapping α of G onto G $\ni \alpha(a) = \iota(a) = a$, $\forall\ a \in G$

$\ni a \neq x$ and $a \neq y$, $\alpha(x) = y$, and $\alpha(y) = x$. Let $U \in \mathcal{J}$. Since $(G,\mathcal{J})$ is T_0, $x \in U \iff y \in U$. And, $\alpha(a) = a$, $\forall a \in U \ni a \neq x$ and $a \neq y$. Hence, $\alpha[U] = U$. Let $T \in \mathcal{J}_{\mathcal{B}}$. Then $\alpha \in T \iff \exists U_1, \ldots, U_n \in \mathcal{J} \ni$ $B_{U_1,U_1} \cap B_{U_2,U_2} \cap \ldots \cap B_{U_n,U_n} \subset T$, which, by Corollary 4, holds if and only if $\iota \in T$. Hence, $(\mathcal{Q}, \mathcal{J}_{\mathcal{B}})$ is not T_2.

Let $B' = B_{U_1,U_2} \cap \ldots \cap B_{U_{n-1},U_n} \in \mathcal{B}$. $\forall i \in \{1,3,\ldots,n-1\}$, let $F_i = \{V \in \mathcal{J} | V \neq U_i\}$ and $\ni \exists \alpha \in \mathcal{Q} \ni \alpha[U_i] = V$. Then $B_{U_i,U_{i+1}} =$ $\mathcal{C}_{V \in F_i} \bigcup B_{U_i,V} = \bigcap_{V \in F_i} \mathcal{C}B_{U_i,V}$, which is an intersection of closed sets and, therefore, closed. Hence, B', itself an intersection of closed sets, is closed. Therefore, B is closed. $\forall B \in \mathcal{B}$.

Let $\alpha \in \mathcal{Q}$ and let $U \in \mathcal{J}_{\mathcal{B}} \ni \alpha \in U$. Then $\exists B \in \mathcal{B} \ni B \subset U$. And, by the foregoing, B is closed. Hence, $\mathcal{J}_{\mathcal{B}}$ is regular.

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