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Topological properties of Hilbert space

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PREFACE

In studying the various topology books and other literature concerning Hilbert Space, it is quite apparent that topologically, it is closely related to Euclidean n-space. Both are metric spaces and although Hilbert Space is not the product of the real line E_1 , taken infinitely many times, its points are defined by infinite sequences of real numbers. Hence, we have that Hilbert Space like Euclidean n-space is also a vector space. Topologically, these two spaces differ in that Hilbert Space is not locally compact, whereas Euclidean n-space is locally compact. In a later theorem, it will be shown that one can always embed Euclidean n-space in Hilbert Space by an isometry. The purpose then, of this thesis, is to develop the topological properties of Hilbert Space.

Hilbert Space obtained its name from the famous German mathematician D. Hilbert (1862-1943). In his works on the theory of integral equations, Hilbert considered some specific spaces composed of sequences (l_2) and of functions, which were later used as models for the construction of the general theory of spaces called by his name.

In 1928, Frechet raised the general question as to which linear topological spaces were homeomorphic to each other. Specifically, he asked whether Hilbert Space was homeomorphic to S, where $S = \prod_{i=1}^{\infty} I_i^o$, where for each $i > 0$, I_i^o denotes the open interval $(0,1)$. In 1932, Banach stated that Mazur had shown that S was not homeomorphic to Hilbert Space. Subsequently it was understood that the question was still open. Between 1932 and 1966, there have been many people working to solve this problem, with notable contributions from Kadec, Bessaga, and Pelczynski. However, the topological classification of separable infinite-dimensional Frechet Spaces was recently completed, when R. D. Anderson, using some important results of Kadec, Bessaga, and Petczynski, has shown that Hilbert Space is homeomorphic to S. Hence, all such spaces are homeomorphic to each other. For a more complete and detailed description of this problem and its very important proof, see Anderson's article, "Hilbert Space is homeomorphic to the countable Infinite Product of Lines," in the Bulletin of the American Mathematical Society, Vol. 72, No. 3, May 1966, pg. 515-19.

Most of the results in this thesis are well known, however the proofs were developed by the author with occasional assistance by his graduate adviser and members of his committee. Where this is not the case, the source has been clearly indicated.

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CHAPTER I

INTRODUCTION

In this chapter we shall introduce Hilbert Space and show that it is a metric space. Then, using the fact that it is a metric space, we will define a topology for the space, hence giving us that Hilbert Space is a topological space. Throughout this chapter, we will be chiefly interested in introducing terms and results that will be of importance to us in the later chapters, where we attempt the more difficult topological properties. We conclude the chapter by showing that Hilbert Space is not the product of the real line, taken with itself, infinitely many times.

DEFINITION 1.1 The set of all sequences $\{x_n\}$ of real numbers for which $\sum_{n=1}^{\infty} x_n^2 < \infty$ and where the distance between two such sequences $\{x_n\}$ and $\{y_n\}$ is defined to be $\left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$ is denoted by E^{ω} and is known as Hilbert Space. Such a sequence $\{x_n\}$ is known as a point in Hilbert Space.

DEFINITION 1.2 A pair of objects (X,ρ) consisting of a non-empty set X and a function $\rho : X \times X \longrightarrow R$ is a metric space provided that:

1. $\rho(x,y) \geq 0$ for all $x,y \in X$.

2. $\rho(x,y) = 0$ if and only if $x = y$ for $x,y \in X$.

3. $\rho(x,y) = \rho(y,x)$ for all $x,y \in X$.

4. $\rho(x,z) \leq \rho(x,y) + \rho(y,z)$ for all $x,y,z \in X$.

<u>THEOREM 1.1</u>

E^{ω} is a metric space.

Proof:

Let $x = \{x_n\}$ and $y = \{y_n\}$ be any two points in E^{ω} . To begin with, we would like to show that

$\rho(x,y) = \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$ is always convergent for all $x,y \in E^{\omega}$. Consider,

$$\sum_{n=1}^{\infty} (x_n - y_n)^2 = \sum_{n=1}^{\infty} (x_n^2 - 2x_ny_n + y_n^2)$$

$$= \sum_{n=1}^{\infty} x_n^2 + \sum_{n=1}^{\infty} y_n^2 - 2 \sum_{n=1}^{\infty} x_ny_n$$

applying the Cauchy Inequality

$$\left(\sum_{n=1}^{\infty} x_ny_n \right)^2 \leq \left(\sum_{n=1}^{\infty} x_n^2 \right)\left(\sum_{n=1}^{\infty} y_n^2 \right)$$

thus getting

$$\sum_{n=1}^{\infty} (x_n - y_n)^2 \leq \sum_{n=1}^{\infty} x_n^2 + \sum_{n=1}^{\infty} y_n^2 + 2\Big(\sum_{n=1}^{\infty} x_n^2\Big)^{1/2}\Big(\sum_{n=1}^{\infty} y_n^2\Big)^{1/2}.$$

Since $x,y \in E^{\omega}$, by definition 1.1 everything on the right hand side of the inequality converges, thus $\sum_{n=1}^{\infty} (x_n - y_n)^2 < \infty$. Therefore, for all $x,y \in E^{\omega}$, $\rho\,(x,y) < \infty$.

Since the point $0 = (0, 0, \ldots)$ is in E^{ω}, E^{ω} is clearly non-empty. Now in calculating $\rho\,(x,y)$, we are always squaring the difference, adding non-negative numbers, and then taking the square root of a non-negative number. Hence, $\rho\,(x,y) \geq 0$ for all $x,y \in E^{\omega}$. Suppose that for $x,y \in E^{\omega}$, we have that $x = y$, then for all n, $x_n = y_n$ and $\Big[\sum_{n=1}^{\infty} (x_n - y_n)^2\Big]^{1/2} = 0$. Thus, $\rho\,(x,y) = 0$. Now suppose that $\rho\,(x,y) = 0$, then $\sum_{n=1}^{\infty} (x_n - y_n)^2 = 0$. Hence, we have in the sum, the infinite series of non-negative numbers is zero, hence, each number of the series must be zero. Thus, for all n, $x_n = y_n$ which implies that $x = y$. Now for each $x,y \in E^{\omega}$, we have that

$$\rho\,(x,y) = \Big[\sum_{n=1}^{\infty} (x_n - y_n)^2\Big]^{1/2} = \Big[\sum_{n=1}^{\infty} (y_n - x_n)^2\Big]^{1/2} = \rho\,(y,x).$$

Now let $x, y, z \in E^{\omega}$, then

$$\rho\ (x,z) = \left[\sum_{n=1}^{\infty} (x_n - z_n)^2\right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} ((x_n - y_n) + (y_n - z_n))^2\right]^{1/2} .$$

If we now square both sides and the terms inside the summation, we will get

$$\left[\rho\ (x,z)\right]^2 = \sum_{n=1}^{\infty} (x_n - y_n)^2 + \sum_{n=1}^{\infty} (y_n - z_n)^2$$

$$+ 2 \sum_{n=1}^{\infty} (x_n - y_n)(y_n - z_n) = \left[\rho\ (x,y)\right]^2$$

$$+ \left[\rho\ (y,z)\right]^2 + 2 \sum_{n=1}^{\infty} (x_n - y_n)(y_n - z_n)$$

again, applying the Cauchy Inequality

$$\sum_{n=1}^{\infty} (x_n - y_n)(y_n - z_n)$$

$$\leq \left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{1/2} \left[\sum_{n=1}^{\infty} (y_n - z_n)^2\right]^{1/2} .$$

Thus, getting that

$$[\rho\ (x,z)]^2 \leq [\rho\ (x,y)]^2 + [\rho\ (y,z)]^2 + 2\,\rho\ (x,y)\,\rho\ (y,z)$$

$$= [\rho\ (x,y) + \rho\ (y,z)]^2$$

$$\rho\ (x,z) \leq \rho\ (x,y) + \rho\ (y,z)$$

Therefore, E^{ω} is a metric space.

<u>DEFINITION 1.3</u> Let X be a non-empty set and J a collection of subsets of X such that

1. X and ϕ are members of J .
2. If $O_1, O_2, \ldots, O_n$ are in J , then $\bigcap_{j=1}^{n} O_j \in J$.
3. If for each $\alpha \in I$, $O_\alpha \in J$, then $\bigcup_{\alpha \in I} O_\alpha \in J$.

The pair of objects (X,J) is called a <u>topological space</u>. The set X is called the <u>underlying set</u> and the collection J is called the <u>topology</u> on the set X . The members of J are called the <u>open sets</u> in the topological space X .

<u>DEFINITION 1.4</u> Let x be some point in E^{ω} and let $r > 0$. An <u>open sphere of radius r</u> about x , denoted by $S\ (x,r)$, will consist of the set of all points at distance less than r from x , that is;

$$S\ (x,r) = \{\ y \mid \rho\ (x,y) < r\ \} .$$

DEFINITION 1.5 A set U in a metric space is open if and only if for each point x in U, there exists an open sphere S (x,r) containing x such that S (x,r) $\subset$ U.

DEFINITION 1.6 A set U in a topological space (X,J) is a neighborhood of a point x if and only if U contains an open set to which x belongs.

DEFINITION 1.7 A family β of open sets is a base for a topology J if and only if, β is a subfamily of J and for each point x of the space, and each neighborhood U of x, there is a member V of β such that $x \in V \subset U$.

We would now like to establish that the set of all open spheres in E^{ω} is a base for the topology in E^{ω}. This topology will consist of the collection of all sets formed by unions and finite intersections of open spheres.

THEOREM 1.2

The family of all open spheres in E^{ω} is a base for the topology J of E^{ω}.

Proof:

Since Hilbert Space is a metric space, it will be sufficient to show that the intersection of two open spheres contains an open sphere about each of its points. Let x and y be two points in E^{ω} and let S (x,r) and S (y,r) be two open spheres about x and y of radius r. Let $z \in S(x,r) \cap S(y,r)$, then

$z \in S(x,r)$ and $z \in S(y,r)$. Now since $S(x,r)$ is open, there is some $\delta_1 > 0$ such that $S(z,\delta_1) \subset S(x,r)$. Likewise, there exists some $\delta_2 > 0$ such that $S(z,\delta_2) \subset S(y,r)$. Let $\delta = \min.(\delta_1, \delta_2)$, then

$$S(z,\delta) \subset S(x,r) \cap S(y,r)$$

Now z being arbitrary implies that this result holds for all points belonging to $S(x,r) \cap S(y,r)$. Hence, by definition, the family of all open spheres in E^{ω} is a base for the topology J of E^{ω}.

We would at this time like to state two properties, seemingly unrelated to the material already presented in this chapter, that will be of considerable importance throughout the remainder of this thesis.

DEFINITION 1.8 The distance from a point x to a non-void subset A of a metric space is defined to be $\rho(x,A) = \inf\{\rho(x,y) \mid y \in A\}$.

LEMMA 1.1

If A is a fixed subset of a metric space, then the distance from a point x to A is a continuous function of x relative to the metric topology. (See appendix for proof.)

We will now conclude the chapter with the following result;

THEOREM 1.3

E^{ω} is not the product space $\times\{S_n \mid S_n = E_1 \text{ for all } n \in \omega\}$.

Proof:

Let $S = \times\{S_n \mid S_n = E_1 \text{ for all } n \in \omega\}$. Let $x = (x_1, x_2, \ldots)$ be any point in E^{ω}, then clearly from the definition of E^{ω} (definition 1.1), $x \in S$. Now consider the set of all points $y_n = (n, n, n, \ldots)$ for all $n \in \omega$, belonging to S.

Since for all $n \in \omega$, $\sum_{n=1}^{\infty} y_{n_i}^2 = \infty$, we have that $y_n \notin E^{\omega}$. Therefore, $E^{\omega} \subset S$.

CHAPTER II

TOPOLOGICAL PROPERTIES

Separation Axioms

This chapter will be devoted to developing the various separation properties that topological spaces may possess. We will begin by showing E^ω to be a T_o-space and then proceed to add on more demanding requirements. The chapter will be concluded by showing that Hilbert Space is actually a Tychonoff Space and is hence homeomorphic to a subspace of a cube.

DEFINITION 2.1 A topological space is a T_o-space if and only if for any two distinct points a and b, there is an open set containing a but not b.

THEOREM 2.1

E^ω is a T_o-space.

Proof:

Let $a = \{a_n\}$ and $b = \{b_n\}$ be any two distinct points of E^ω, then there exists some $r > 0$ such that

$$\rho\ (a,b) = \left[\sum_{n=1}^{\infty} (a_n - b_n)^2 \right]^{1/2} = 3r$$

Now construct an open sphere $S(a,r)$ about the point a of radius r, then certainly $a \in S(a,r)$ but $b \notin S(a,r)$. Thus, E^ω is a T_0-space.

<u>DEFINITION 2.2</u> A topological space X is a <u>T_1-space</u> if and only if each set which consists of a single point is closed, that is, $\{x\}$ is closed for all $x \in X$.

<u>THEOREM 2.2</u>

E^ω is a T_1-space.

Proof:

Let $x = \{x_n\}$ be any point of E^ω. We wish to show that $\{x\}$ is closed. Consider $E^\omega - \{x\}$ and let $y \in (E^\omega - \{x\})$. Let $\rho(x,y) = 3r$, then we can construct an open sphere $S(y,r)$ about y such that $x \notin S(y,r) \subset (E^\omega - \{x\})$. Hence, $E^\omega - \{x\}$ is a neighborhood of all of its points, which implies that $E^\omega - \{x\}$ is open. Therefore, $\{x\}$ is closed and E^ω is a T_1-space.

<u>DEFINITION 2.3</u> A topological space is <u>Hausdorff</u>, T_2, if and only if whenever x and y are distinct points of the space, there exist disjoint neighborhoods of x and y.

<u>THEOREM 2.3</u>

E^ω is a Hausdorff Space.

Proof:

Suppose that $x = \{x_n\}$ and $y = \{y_n\}$ are distinct points of E^ω, then there is some $r > 0$ such that $\rho(x,y) = 3r$. Construct open spheres S_1 and S_2 about x and y of radius r. Then $x \in S_1$ and $y \in S_2$ and $S_1 \cap S_2 = \phi$. To show this, let a be any point belonging to S_1, then $\rho(x,a) < r$ and

$$\rho(x,a) + \rho(a,y) \geq \rho(x,y)$$
$$\rho(a,y) \geq \rho(x,y) - \rho(x,a)$$
$$\rho(a,y) > 3r - r = 2r$$

Hence $a \notin S_2$ and E^ω is Hausdorff.

<u>DEFINITION 2.4</u> A topological space is <u>regular</u>, if and only if for each point x and each neighborhood U of x, there is a closed neighborhood V of x such that $V \subset U$.

<u>THEOREM 2.4</u>

E^ω is regular.

Proof:

Let x be any point of E^ω, and let U be any neighborhood of x. Then for some $r > 0$, we can construct an open sphere $S(x,r)$ such that $x \in S(x,r) \subset U$. Let $T = \{y \mid y \in E^\omega, \rho(x,y) \leq r/2\}$, then T is closed, since $\rho(x,y)$ is continuous and the set of points where $\rho(x,y) \leq r$ is closed. Therefore, $x \in T \subset S(x,r) \subset U$.

Hence E^{ω} is a regular space.

DEFINITION 2.5 A topological space is normal if and only if for each disjoint pair of closed sets, A and B, there are disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

THEOREM 2.5

E^{ω} is normal.

Proof:

Let A and B be disjoint closed sets in E^{ω}. For each $x \in E^{\omega}$, let $\rho(A,x)$ and $\rho(B,x)$ be the distance from x to A and from x to B. Let

$$f(x) = \rho(A,x) - \rho(B,x)$$

then f is continuous in x. Now let

$$U = \{x \mid f(x) < 0\} \quad \text{and} \quad V = \{x \mid f(x) > 0\}$$

then U and V are open and disjoint, since f is continuous. The closure of a set in a metric space is the set of all points which are zero distance from the set. Hence, $A \subset U$ and $B \subset V$, and E^{ω} is normal.

Now since a regular T_1-space is a T_3-space and a normal T_1-space is a T_4-space, we have that Hilbert Space is also T_3 and T_4.

<u>LEMMA 2.1</u> (Urysohn)

Let X be a normal space, A, B closed disjoint subsets of X, then there exists a continuous mapping $f : X \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$ (for proof see appendix).

<u>DEFINITION 2.6</u> A topological space X is called <u>completely regular</u> if and only if for each $x \in X$ and each neighborhood U of x, there is a continuous function f on X to $[0,1]$ such that $f(x) = 0$ and $f(X - U) = 1$.

<u>THEOREM 2.6</u>

E^{ω} is a completely regular space.

Proof:

Let x be any point of E^{ω} and U an open neighborhood of x in E^{ω}. By theorem 2.2, E^{ω} is a T_1-space, hence, $\{x\}$ is a closed set in E^{ω}. Let $A = \{x\}$ and $B = (E^{\omega} - U)$, then A and B are disjoint closed subsets of E^{ω}. By theorem 2.5, E^{ω} is normal. Hence, lemma 2.1 says that there exists a continuous function $f : E^{\omega} \longrightarrow [0,1]$ such that $f(A) = 0$ and $f(B) = 1$. Therefore, E^{ω} is a completely regular space.

The cartesian product of closed unit intervals, with the product topology, is called a <u>cube</u>. A cube is then the set Q^A of all functions of a set A to the closed unit interval Q, with the topology of pointwise, or coordinate-wise, convergence. Suppose that F is a family of functions such that each member f of F is on a topological space X to a space Y_f (the range may be different for different members of F). There is then a natural mapping of X into the product $\times\{Y_f \mid f \in F\}$ which is defined by mapping a point x of X into the member of the product whose f-th coordinate is $f(x)$. Formally, the <u>evaluation map</u> e is defined by: $e(x)_f = f(x)$. It turns out that e is continuous if the members of F are continuous and e is a homeomorphism if, in addition, F contains "enough functions".

<u>DEFINITION 2.7</u> A family of functions F on X <u>distinguishes points</u> if and only if for each pair of distinct points x and y there is an $f \in F$ such that $f(x) \neq f(y)$. The family F <u>distinguishes points and closed sets</u> if and only if for each subset A of X and each $x \in X - A$, there is an $f \in F$ such that $f(x)$ does not belong to the closure of $f(A)$.

<u>EMBEDDING LEMMA 2.2</u>

Let F be a family of continuous functions each member f being on a topological space X into a topological space Y_f. Then

(a) The evaluation map e is a continuous function on X to the product space $\times\{Y_f \mid f \in F\}$.

(b) The function e is an open map of X onto $e[X]$ if F distinguishes points and closed sets.

(c) The function e is one-to-one if and only if F distinguishes points.

(for proof see appendix)

Now since a completely regular T_1-space is a Tychonoff Space, we see that E^ω is a Tychonoff Space. If we let F be the family of all continuous functions on E^ω to $[0,1]$, then the Embedding Lemma 2.2, shows that the evaluation map of E^ω into the cube Q^F is a homeomorphism, since

EMBEDDING THEOREM 2.7

In order that a topological space be a Tychonoff Space it is necessary and sufficient that it be homeomorphic to a subspace of a cube. (for proof see appendix)

In this chapter we have shown that Hilbert Space possesses the important separation properties. We concluded by showing E^ω to be a Tychonoff Space, and hence by the Embedding Theorem 2.7, it may be mapped into a cube by a homeomorphism.

CHAPTER III

TOPOLOGICAL PROPERTIES

In this chapter we shall be interested in showing that Hilbert Space is a complete separable metric space, but is neither compact nor locally compact. After establishing these properties, we will then introduce what is known as the Hilbert Cube and show that it is compact and that no non-empty open subset of E^{ω} is contained in the Hilbert Cube. However, first we would like to look at an important result concerning convergence of infinite sequences in E^{ω} .

<u>THEOREM 3.1</u>

Given $x_n = (\alpha_{1n}, \alpha_{2n}, \alpha_{3n}, . . .)$ a sequence of points in E^{ω} , $\{x_n\}$ converges to $a = (a_1, a_2, a_3, . . .)$ then $\lim_{n \to \infty} \alpha_{kn} = a_k$ and this point a is unique.

Proof:

Given $r > 0$, let $\{x_n\}$ be a sequence of points in E^{ω} converging to $a = (a_1, a_2, a_3, . . .)$. Then there exists an integer $N > 0$, such that for $n > N$,

$$\rho\ (x_n, a) = \left[\sum_{k=1}^{\infty} (\alpha_{kn} - a_k)^2 \right]^{1/2} < r$$

holds for all k . Then for each k , we have that $(\alpha_{kn} - a_k)^2 < r^2$ or that $|\alpha_{kn} - a_k| < r$. Hence $\lim_{n \to \infty} \alpha_{kn} = a_k$.

Now suppose that $\lim_{n \to \infty} \alpha_{kn} = a_k$ holds for all k and $\{x_n\}$ converges to $b = (b_1, b_2, b_3, \ldots)$. Now from the first part of the theorem, we see that for $r > 0$, there is an $N > 0$, such that for $n > N$, $|\alpha_{kn} - b_k| < r$ for all k . Then $\lim_{n \to \infty} \alpha_{kn} = b_k$ for each k , But, by assumption, we have that $\lim_{n \to \infty} \alpha_{kn} = a_k$, and since a sequence of real numbers can converge to at most one point, we have that $a_k = b_k$ for all values of k .

We would now like to present an example showing that the converse of theorem 3.1 does not hold in E^ω . That is, if a sequence of points belongs to E^ω , then it is not necessarily true that the sequence will always converge to a point of E^ω .

<u>EXAMPLE:</u> Consider the following collection of points

$$x_0 = \left[1, \ \frac{1}{2}, \ \frac{1}{3}, \ \frac{1}{4}, \ \ldots\right]$$

$$x_1 = \left[1, \ \frac{1}{(2)^{3/4}}, \ \frac{1}{(3)^{3/4}}, \ \frac{1}{(4)^{3/4}}, \ \ldots\right]$$

$$x_2 = \left[1, \ \frac{1}{(2)^{5/8}}, \ \frac{1}{(3)^{5/8}}, \ \frac{1}{(4)^{5/8}}, \ \ldots\right]$$

$$\vdots$$

where for each n, the exponent is determined by the formula $\frac{2^n + 1}{2^{n+1}}$. Now for all n, $x_n \in E^\omega$ since

$$\sum_{n=1}^{\infty} \frac{1}{n^{2p}} < \infty \quad \text{if and only if} \quad 2p > 1 .$$

However, since $\frac{2^n + 1}{2^{n+1}} = \frac{1 + 1/2^n}{2}$ we see that

$$\lim_{n \to \infty} \left[\frac{2^n + 1}{2^{n+1}}\right] = \lim_{n \to \infty} \left[\frac{1 + 1/2^n}{2}\right] = \frac{1}{2} .$$

Hence $\lim_{n \to \infty} \alpha_{kn} = \frac{1}{k^{1/2}}$, but $\sum_{k=1}^{\infty} \left[\frac{1}{k^{1/2}}\right]^2 = \infty$ and so the convergence of the coordinates in E^ω does not imply that the sequence of points converges to a point of E^ω.

<u>DEFINITION 3.1</u> Let (X,ρ) be a metric space. Let $\{a_n\}$ be a sequence of points of X. A point $x \in X$ is said to be the <u>limit of the sequence $\{a_n\}$</u> if

$$\lim_{n \to \infty} \rho(a_n, x) = 0 .$$

<u>DEFINITION 3.2</u> In a metric space (X,ρ), a sequence $\{a_n\}$ of points of X is called a <u>Cauchy Sequence</u> if and only if for each $r > 0$, there is an $N > 0$ such that $\rho(a_n, a_m) < r$ whenever $n,m > N$.

<u>DEFINITION 3.3</u> A metric space X is <u>complete</u> if every Cauchy Sequence of points of X converges to a point of X.

<u>THEOREM 3.2</u>

E^{ω} is a complete metric space.

Proof:

For $x_n = (\alpha_{1n}, \alpha_{2n}, \alpha_{3n}, \ldots)$ let $\{x_n\}$ be any Cauchy Sequence of points in E^{ω}. Then for each $r > 0$, there is an $N_r > 0$, such that if $n,m > N_r$

$$[\rho(x_n, x_m)]^2 = \sum_{k=1}^{\infty} (\alpha_{kn} - \alpha_{km})^2 < r .$$

Hence, for each k, $(\alpha_{kn} - \alpha_{km})^2 < r$ if $n,m > N_r$. Thus for each k, the sequence of real numbers $\{\alpha_{kn}\}$ converges. Now since E_1 is complete, there exists an

a_k such that $\lim_{n \to \infty} \alpha_{kn} = a_k$. Let $a = (a_1, a_2, a_3, \ldots)$, then we wish to show

$$(1). \quad \sum_{k=1}^{\infty} a_k^2 < \infty$$

$$(2). \quad \lim_{n \to \infty} \rho\,(x_n, a) = 0$$

Let M be arbitrary and write the inequality as follows;

$$\sum_{k=1}^{\infty} (\alpha_{kn} - \alpha_{km})^2 = \sum_{k=1}^{M} (\alpha_{kn} - \alpha_{km})^2 + \sum_{k=M+1}^{\infty} (\alpha_{kn} - \alpha_{km})^2 < r$$

Since each of these sums is non-negative, each of them is less than r. Consequently, $\sum_{k=1}^{M} (\alpha_{kn} - \alpha_{km})^2 < r$.

If we take the limit as $n \to \infty$, we obtain

$$\sum_{k=1}^{M} (a_k - \alpha_{km})^2 \leq r$$

Since this inequality is valid for arbitrary M, we can take the limit as $M \to \infty$ obtaining $\sum_{k=1}^{\infty} (a_k - \alpha_{km})^2 \leq r$.

$$\lim_{M \to \infty} \rho\,(x_m, a) = \lim_{M \to \infty} \left[\sum_{k=1}^{\infty} (a_k - \alpha_{km})^2 \right]^{1/2} = 0$$

Therefore, from the inequality obtained and the convergence of the series $\sum_{k=1}^{\infty} \alpha_{km}^2$ we can get the following results;

$$(2\alpha_{km} - a_k)^2 \geq 0$$

$$4\alpha_{km}^2 - 4\alpha_{km}a_k + a_k^2 \geq 0$$

$$4\alpha_{km}^2 - 4\alpha_{km}a_k + 2a_k^2 \geq a_k^2$$

$$2\left[\alpha_{km}^2 - 2\alpha_{km}a_k + a_k^2\right] + 2\alpha_{km}^2 \geq a_k^2$$

$$2\sum_{k=1}^{\infty} (\alpha_{km} - a_k)^2 + 2\sum_{k=1}^{\infty} \alpha_{km}^2 \geq \sum_{k=1}^{\infty} a_k^2$$

and since everything on the left hand side converges, we have that $\sum_{k=1}^{\infty} a_k^2$ converges. Therefore, $a = \{a_k\}$ is a point in E^{ω} and Hilbert Space is complete.

<u>DEFINITION 3.4</u> A set A is <u>dense</u> in a topological space if and only if the closure of A is the whole space.

<u>DEFINITION 3.5</u> A topological space is <u>separable</u> if and only if there is a countable subset which is dense in the space.

<u>THEOREM 3.3</u>

E^{ω} is a separable space.

Proof:

We will define a point of E^ω to be rational if each coordinate of the point is a rational number. Let $x = (x_1, x_2, x_3, \ldots)$ be any element in E^ω. For each coordinate of x, there is a countable number of rational numbers as entries, and since there are a countable number of coordinates of x, the rational points of E^ω are countable. Consider, the set of all points T such that all but a finite number of the coordinates are zero, and those that are not zero are rational. Given $r > 0$, let $x = \{x_n\}$ be any point in E^ω. Choose N so large that $\sum_{n=N}^{\infty} (x_n)^2 < r^2/2$. Now consider all points $a = (a_1, a_2, a_3, \ldots)$ in T such that for $n \geq N$, $a_n = 0$. We wish to show that we can find a member of T as close to $x = \{x_n\}$ as we choose. For $n = 1, 2, \ldots, N-1$, choose a_n such that $| a_n - x_n | < r/[2(N-1)]^{1/2}$. This is always possible since the rational numbers are dense in E_1, then

$$\rho\ (x,a) = \left[\sum_{n=1}^{\infty} (a_n - x_n)^2 \right]^{1/2}$$

$$= \left[\sum_{n=1}^{N-1} (a_n - x_n)^2 + \sum_{n=N}^{\infty} (a_n - x_n)^2 \right]^{1/2}$$

$$= \left[\sum_{n=1}^{N-1} (a_n - x_n)^2 + \sum_{n=N}^{\infty} (x_n)^2 \right]^{1/2}$$

$$< \left[\sum_{n=1}^{N-1} r^2/2(N-1) + r^2/2 \right]^{1/2} = r$$

Therefore, the rational numbers in E^{ω} are a countable dense subset of E^{ω} and this implies that E^{ω} is separable.

Having now established Hilbert Space complete and separable, we would like to show that it is neither compact nor locally compact, hence, producing a major topological difference between E^{ω} and Euclidean n-space. We will introduce two different but equivalent definitions of compactness, the first of which is;

<u>DEFINITION 3.6</u> A family $\mathcal{A}$ is a <u>cover</u> of a set B if and only if B is contained in $\bigcup \{A \mid A \in \mathcal{A}\}$. The family $\mathcal{A}$ is an <u>open cover</u> of B if and only if each member of $\mathcal{A}$ is an open set. A is a <u>subcover</u> of $\mathcal{A}$ if and only if $A \subset \mathcal{A}$.

<u>DEFINITION 3.7</u> A topological space X is <u>compact</u> if and only if each open cover of X has a finite subcover.

<u>THEOREM 3.4</u>

E^{ω} is not compact.

Proof:

Consider the following sequence of points

$$
\begin{aligned}
x_1 &= (1, 0, 0, \ldots, 0, \ldots) \\
x_2 &= (0, 2, 0, \ldots, 0, \ldots) \\
x_3 &= (0, 0, 3, \ldots, 0, \ldots) \\
&\vdots \\
x_n &= (0, 0, 0, \ldots, n, \ldots) \\
&\vdots
\end{aligned}
$$

characterized by the fact that all coordinates are zero except the n-th coordinate of x_n, which is n. Clearly, for all values of n, x_n is in E^{ω}. Now for all n, let A_n be an open sphere of radius $r = 1$ containing the point x_n. Then $\{A_n\}$, $n = 1, 2, \ldots$ is an open covering of $\{x_n\}$. However, the distance between two points x_n and x_m of $\{x_n\}$, where $n \neq m$, is always

$$\rho\ (x_n,x_m) = [n^2 + m^2]^{1/2} > 1$$

for all m and n . Hence, each member of $\{A_n\}$ can contain at most one member of $\{x_n\}$ and since $\{x_n\}$ is infinite, no finite subcoverings of $\{x_n\}$ by members of $\{A_n\}$ will cover $\{x_n\}$. Therefore, E^ω is not compact.

DEFINITION 3.8 A topological space X is said to be locally compact provided that for each $x \in X$, there exists at least one compact neighborhood of x .

DEFINITION 3.9 A subset A of a topological space S is said to be countably compact if and only if every infinite subset of A has at least one limit point in A .

LEMMA 3.1

A subset T of a metric space S is compact if and only if it is countably compact.
(for proof see appendix)

THEOREM 3.5

E^ω is not locally compact.

Proof:

Suppose E^ω is locally compact. Then there is a compact neighborhood U of the origin $0 = (0, 0, 0, \ldots)$. Choose $r > 0$, such that the closed sphere of radius r about the origin is included in U. Let

$$\delta = \frac{r}{\left[\sum_{n=1}^{\infty} 1/n^2 \right]^{1/2}}$$

$$(b_{ij}) = \begin{cases} 0 & , \quad j > i \geq 1 \\ \dfrac{\delta}{i - j + 1} & , \quad i \geq j \geq 1 \end{cases}$$

let $B_j = (b_{ij})$, then we get the following family of sequences

$$B_1 = (\delta, \delta/2, \delta/3, \ldots, \delta/k, \ldots)$$
$$B_2 = (0, \delta, \delta/2, \ldots, \delta/k-1, \ldots)$$
$$B_3 = (0, 0, \delta, \ldots, \delta/k-2, \ldots)$$
$$\vdots$$
$$B_k = (0, 0, 0, \ldots, \delta, \delta/2, \ldots)$$
$$\vdots$$

Now from this family of sequences that we have just defined, we can get the following results:

(i) $\rho\ (0,B_j) = \left[\sum_{i=1}^{\infty} (0 - b_{ij})^2 \right]^{1/2}$

$$= \delta \left[\sum_{i=1}^{\infty} 1/i^2 \right]^{1/2} = r$$

holds for each j .

(ii) $\lim_{j \to \infty} b_{ij} = 0$ holds for all i , hence

$B_j \to 0$ as $j \to \infty$.

But the compactness of U requires that the sequence $\{B_j\}$ must have a convergent subsequence, see definition 3.9 and lemma 3.1. Also, the convergence of $\{B_j\}$ requires that the sequence $\{B_j\}$ must converge to the origin, see definition 3.1. But by (i), we have that no subsequence can converge to the origin. A contradiction to the hypothesis that E^ω is locally compact. Hence, E^ω is not locally compact.

Having now established that E^ω is neither compact nor locally compact, we would like to introduce the most important subspace of E^ω . This subspace is called the Hilbert Cube. We will show that it is compact, hence locally compact, since a compact space is automatically locally compact.

<u>DEFINITION 3.10</u> The subspace of E^{ω} consisting of all sequences $\{x_n\}$ such that $|x_n| \leq 1/n$ is denoted by J^{ω} and is called the <u>Hilbert Cube</u>.

<u>THEOREM 3.6</u>

The Hilbert Cube, J^{ω} , is compact.

Proof:

Let $x_n = (z_{1n}, z_{2n}, z_{3n}, \ldots)$ represent any arbitrary sequence of points such that each $x_n \in J^{\omega}$. We wish to show that $\{x_n\}$ contains a subsequence which converges to a point of J^{ω} . Consider the sequence formed by the first coordinates of the points of $\{x_n\}$. Doing this we get

$$P_1 = (z_{11}, z_{12}, z_{13}, \ldots)$$

where $|z_{1n}| \leq 1$ for all n . Now every infinite bounded set in E_1 has at least one limit point in E_1 . Hence, there exists some a_1 such that $|a_1| \leq 1$ and a subsequence

$$P_1' = (z_{1\alpha_1}, z_{1\alpha_2}, z_{1\alpha_3}, \ldots)$$

which converges to a_1 . Now form the sequence of the second coordinates from the members of $\{x_n\}$ whose first coordinates define P_1'. We will then get

$$P_2 = (z_{2\alpha_1}, z_{2\alpha_2}, z_{2\alpha_3}, \ldots)$$

where $|z_{2\alpha_i}| \leq 1/2$ for all values of i. Thus, there is some a_2 such that $|a_2| \leq 1/2$ and a subsequence

$$P_2' = (z_{2\beta_1}, z_{2\beta_2}, z_{2\beta_3}, \ldots)$$

which converges to a_2. Continuing this process k times, we will get a sequence of points

$$P_k = (z_{k\rho_1}, z_{k\rho_2}, z_{k\rho_3}, \ldots)$$

that are the k-th coordinates of the members of $\{x_n\}$ whose (k-1)-th coordinates converged to some point a_{k-1} such that $|a_{k-1}| \leq 1/k-1$. Similarily, since for all i, $|z_{k\rho_i}| \leq 1/k$, there exists some a_k such that $|a_k| \leq 1/k$ and a subsequence

$$P_k' = (z_{k\mu_1}, z_{k\mu_2}, z_{k\mu_3}, \ldots)$$

which converges to a_k. Therefore, for all n, we can find a sequence of points which contains a subsequence whose n-th coordinates converge to a point a_n such that $|a_n| \leq 1/n$.

We wish to find a subsequence of $\{x_n\}$ and show that it converges to a point of J^ω . Let us construct a subsequence as follows: choose the point x_{α_1} whose first coordinate is $z_{1\alpha_1}$, x_{β_2} whose first coordinate is $z_{1\alpha_2}$ and second coordinate is $z_{2\beta_2}$, continuing this process, we see that the k-th point x_{μ_k} would have coordinates $(z_{1\alpha_k}, z_{2\beta_k}, \ldots, z_{k\mu_k}, \ldots)$. Thus, we can construct a subsequence of $\{x_n\}$

$$(x_{\alpha_1}, x_{\beta_2}, \ldots, x_{\mu_k}, \ldots)$$

which certainly converges to $a =$ $(a_1, a_2, a_3, \ldots, a_k, \ldots)$ and since k was arbitrary, we know that it holds for all k . Also, for all n , $|a_n| \leq 1/n$ which implies that $a \in J^\omega$. Therefore, J^ω is countably compact and hence by lemma 1.1, J^ω is compact.

<u>THEOREM 3.7</u>

J^ω contains no non-empty open subset of E^ω .

Proof:

Let T be some non-empty open subset of E^{ω}. Suppose that $T \subset J^{\omega}$ and let $x = (x_1, x_2, x_3, \ldots)$ be some point of T. Now since T is open, for some $r > 0$, there is an open sphere about x of radius r such that

$$S(x,r) \subset T \subset J^{\omega}$$

Now consider the point $y = (x_1, x_2, \ldots, x_n + r/2, x_{n+1}, \ldots)$. Clearly $y \in S(x,r)$. Now if we go out far enough so that $1/n < r/4$ then

$$-r/4 \leq x_n \leq r/4$$

$$0 < r/4 \leq x_n + r/2 \leq 3r/4$$

hence $$1/n < r/4 \leq |x_n + r/2|$$

which implies that $y \notin J^{\omega}$. Therefore, T cannot be a subset of J^{ω}.

CHAPTER IV

TOPOLOGICAL PROPERTIES

Our chief aim in this chapter will be to establish that Hilbert Space is both locally connected and connected. To do this we will begin by introducing the notion of a line segment in E^{ω}, and then proceed to show that every line segment in E^{ω} is connected. This last statement will be the result of showing that every line segment in E^{ω} can be mapped onto the unit interval $[0,1]$ by a homeomorphism. Thus, since homeomorphisms preserve topological properties, and since $[0,1]$ is connected, we will easily establish that all line segments in E^{ω} are connected. The chapter will be concluded by showing that both E^{ω} and J^{ω} are perfect sets.

DEFINITION 4.1 Let x and y be any two distinct points in E^{ω}. A line segment between x and y will be given by

$$L\,(x,y) = [z = \alpha x + (1 - \alpha)y \mid 0 \leq \alpha \leq 1]$$

LEMMA 4.1

Let $L\,(x,y)$ be any line segment in E^{ω}. For each $z \in L\,(x,y)$ the following holds

$$(1) \quad \rho\,(x,z) + \rho\,(z,y) = \rho\,(x,y)$$

(2) There is one and only one α such that $z = \alpha x + (1 - \alpha)y$.

Proof:

Let $x = \{x_n\}$ and $y = \{y_n\}$ be two points in E^ω , and let $L(x,y)$ be a line segment between them. Let $z = \{z_n\}$ be any point belonging to $L(x,y)$, then

$$\rho(x,z) + \rho(z,y) = \Big[\sum_{n=1}^{\infty}(x_n - z_n)^2\Big]^{1/2} + \Big[\sum_{n=1}^{\infty}(z_n - y_n)^2\Big]^{1/2}$$

$$= \Big[\sum_{n=1}^{\infty}(x_n - \alpha x_n - (1-\alpha)y_n)^2\Big]^{1/2}$$

$$+ \Big[\sum_{n=1}^{\infty}(\alpha x_n + (1-\alpha)y_n - y_n)^2\Big]^{1/2}$$

where $z_n = \alpha x_n + (1-\alpha)y_n$

$$= \Big[\sum_{n=1}^{\infty}((1-\alpha)x_n - (1-\alpha)y_n)^2\Big]^{1/2}$$

$$+ \Big[\sum_{n=1}^{\infty}(\alpha x_n - \alpha y_n)^2\Big]^{1/2}$$

$$= (1-\alpha)\Big[\sum_{n=1}^{\infty}(x_n - y_n)^2\Big]^{1/2}$$

$$+ \alpha\Big[\sum_{n=1}^{\infty}(x_n - y_n)^2\Big]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$$

$$= \rho\ (x,y)$$

Now suppose that $\rho\ (x,y) = r$. Let $z = \alpha x + (1-\alpha)y$ and $w = \alpha_o x + (1-\alpha_o)y$ be two distinct points on $L\ (x,y)$, then

$$\rho\ (z,w) = \left[\sum_{n=1}^{\infty} (z_n - w_n)^2 \right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} (\alpha x_n + (1-\alpha)y_n - \alpha_o x_n - (1-\alpha_o)y_n)^2 \right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} ((\alpha - \alpha_o)x_n - (\alpha - \alpha_o)y_n)^2 \right]^{1/2}$$

$$= |\ \alpha - \alpha_o\ |\ \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$$

hence, $0 < |\ \alpha - \alpha_o\ | = (1/r)\ \rho\ (z,w) < 1$. Therefore, for distinct points on $L\ (x,y)$, we get distinct values of α .

<u>DEFINITION 4.2</u> Let S and T be topological spaces. <u>S is homeomorphic to T</u> if and only if there is a one-to-one continuous mapping f of S onto T such that f^{-1} is continuous.

LEMMA 4.2

Let x and y be distinct points in E^{ω} and $L(x,y)$ be the line segment between them. For $z = \alpha x + (1-\alpha)y$, the mapping $f : L(x,y) \longrightarrow [0,1]$ defined by

$$f(x) = 1$$
$$f(y) = 0$$
$$f(z) = \alpha \quad \text{for all} \quad z \in L(x,y)$$

defines a homeomorphism between $L(x,y)$ and $[0,1]$ in E_1.

Proof:

Obviously, from the uniqueness of α, the mapping will be one-to-one and onto. We now wish to verify that f and f^{-1} are both continuous. We will first show that f is continuous at an arbitrary point $z \in L(x,y)$, hence it will be continuous for all points of the line segment. Let $\rho(x,y) = r$ then given $\sigma > 0$, let $\delta = r\sigma$ and suppose that $z = \{z_n\}$ and $w = \{w_n\}$ are points of $L(x,y)$ such that $\rho(z,w) < \delta$, then

$$\rho(z,w) = \left[\sum_{n=1}^{\infty} (z_n - w_n)^2\right]^{1/2}$$

$$= |\alpha - \alpha_o| \left[\sum_{n=1}^{\infty} (x_n - y_n)^2\right]^{1/2} < \delta$$

hence, $|\alpha - \alpha_o| < \delta/r$. Therefore,

$| f(z) - f(w) | = | \alpha - \alpha_o | < \delta/r = \sigma$ which implies that f is continuous at z. Hence, f is continuous at all points of $L(x,y)$.

Now, to show f^{-1} continuous. For $\sigma > 0$, choose $\delta = \sigma/r$. Then if $| \alpha - \alpha_o | < \delta$, letting $f^{-1}(\alpha) = z$ and $f^{-1}(\alpha_o) = w$ we have that

$$\rho(f^{-1}(\alpha), f^{-1}(\alpha_o)) = \rho(z,w) = \left[\sum_{n=1}^{\infty} (z_n - w_n)^2 \right]^{1/2}$$

$$= | \alpha - \alpha_o | \left[\sum_{n=1}^{\infty} (x_n - y_n)^2 \right]^{1/2}$$

$$< \delta r = \sigma$$

which implies that f^{-1} is continuous at α, hence continuous at all points of $[0,1]$ since α was arbitrary. Therefore, f is a homeomorphism between $L(x,y)$ and $[0,1]$.

DEFINITION 4.3 A point x is an accumulation point of a subset A of a topological space X if and only if every neighborhood of x contains points of A other than x.

DEFINITION 4.4 A subset A of a topological space X is closed if and only if it contains the set of its accumulation points. The closure of the subset A will be denoted by $\bar{A}$, and is A together with all of its accumulation points.

DEFINITION 4.5 Two subsets A and B are separated in a topological space X if and only if $\bar{A} \cap B$ and $A \cap \bar{B}$ are both void.

DEFINITION 4.6 A topological space X is connected if and only if X is not the union of two non-void separated subsets of X .

LEMMA 4.3

All line segments in E^{ω} are connected. If L (x,y) is a line segment between two points x and y , where x and y are points of some open sphere, then L (x,y) is contained in the open sphere.

Proof:

Let L(x,y) be any line segment in E^{ω} . By lemma 4.2, we can define a homeomorphism between L (x,y) and [0,1] . Since [0,1] is connected and homeomorphisms preserve topological properties, L (x,y) is connected.

Now let $p = \{p_n\}$ be any point of E^{ω} and S (p,r) be any open sphere of radius r about p . Suppose that x and y are points in S (p,r) and L (x,y) the line segment between them. Let z be any point on L (x,y) , then

$$\rho\ (p,z) = \left[\sum_{n=1}^{\infty} (p_n - z_n)^2\right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} (p_n - \alpha x_n - (1-\alpha)y_n)^2\right]^{1/2}$$

$$= \left[\sum_{n=1}^{\infty} (\alpha(p_n - x_n) + (1-\alpha)(p_n - y_n))^2\right]^{1/2}$$

$$\leq \alpha \left[\sum_{n=1}^{\infty} (p_n - x_n)^2\right]^{1/2} + (1-\alpha)\left[\sum_{n=1}^{\infty} (p_n - y_n)^2\right]^{1/2}$$

$$= \alpha\rho\ (p,x) + (1-\alpha)\rho\ (p,y)$$

$$\leq \alpha r + (1-\alpha)r = r\ .$$

Hence, $z \in S\ (p,r)$ for all $z \in L\ (x,y)$, thus $L\ (x,y) \subset S\ (p,r)$.

<u>DEFINITION 4.7</u> A topological space X is said to be <u>locally connected at a point p</u> if and only if given any neighborhood U of p , there exists a connected neighborhood V of p such that $V \subset U$. The space X is said to be locally connected if and only if it is locally connected at each of its points.

Having now shown that all line segments in Hilbert Space are connected, we will employ this property of line segments in E^{ω} to show that E^{ω} is a locally connected space, from which connectedness of E^{ω} follows very readily.

THEOREM 4.1

E^{ω} is a locally connected space.

Proof:

Let p be any point in E^{ω} and U be some neighborhood of p. For some $r > 0$, we can find an open sphere $S(p,r)$ such that $p \in S(p,r) \subset U$. Suppose $S(p,r)$ is not connected, then there are disjoint separated open sets A and B of $S(p,r)$ such that $A \cup B = S(p,r)$. Let A be such that $p \in A$ and $q \in B$. Let $L(p,q)$ be the line segment between p and q. Let $C = A \cap L(p,q)$ and $D = B \cap L(p,q)$. Then $C \cap D = \phi$ and no point of C can be a limit point of D. Likewise, no point of D can be a limit point of C, since A and B are disjoint and separated. However, $L(p,q) = C \cup D$ which implies that $L(p,q)$ must not be connected, a contradiction by lemma 4.3. Hence, $S(p,r)$ must be connected and since open spheres in E^{ω} are neighborhoods, we have that E^{ω} is locally connected.

LEMMA 4.4

Let $\mathcal{B}$ be a family of connected subsets of a topological space. If no two members of $\mathcal{B}$ are separated, then $\cup \{B \mid B \in \mathcal{B}\}$ is connected.

(for proof see appendix)

THEOREM 4.2

E^{ω} is a connected space.

Proof:

Let T be the family of all open spheres centered at the origin in E^{ω}. That is,

$T = \left\{S\ (0,r) \mid r > 0 \text{ and } 0 = (0, 0, 0, \ldots)\right\}$.

By theorem 4.1, each member of T is connected and since for $r_1 < r_2$, we have that

$$S\ (0,r_1) \subset S\ (0,r_2)$$

Thus, no two members of T are separated. Therefore, by lemma 4.4, we know that $\cup \left\{S\ (0,r) \mid S\ (0,r) \in T\right\}$ is connected, and since the union of all open spheres centered at the origin is the space E^{ω}, that is

$$\cup \left\{S\ (0,r) \mid S\ (0,r) \in T\right\} = E^{\omega}$$

we have that E^{ω} is connected.

Having now established that E^{ω} is both locally connected and connected, we would now like to investigate the possibility of E^{ω} and J^{ω} being perfect sets.

DEFINITION 4.8 A set S is dense in itself if and only if each point of S is a limit point of S .

DEFINITION 4.9 A closed set which is dense in itself is called perfect.

THEOREM 4.3

E^{ω} is a perfect set.

Proof:

Let $y = (y_1, y_2, y_3, \ldots)$ be any point of E^{ω} and $S(y,r)$ be an open sphere of radius r about y . Consider the following family of points

$$\{x_n\} = (y_1, y_2 + r/2^n, y_3, \ldots)$$

Each of these points is a point of E^{ω} and $\rho(y,x_n) < r$ for all n . Hence, for all r , there is at least a point of E^{ω} other than y in $S(y,r)$. Thus, since y was arbitrary, each point of E^{ω} is a limit point of E^{ω} . Therefore, E^{ω} is closed and dense in itself which implies that E^{ω} is a perfect set.

THEOREM 4.4 (Borel-Lebesgue Theorem)

A set K is closed and bounded if and only if it is compact.

(for proof see appendix)

THEOREM 4.5

J^{ω} is a perfect set.

Proof:

By theorem 3.6, J^{ω} is compact, hence by theorem 4.4, J^{ω} is a bounded closed subset of E^{ω}. Let $y = (y_1, y_2, y_3, \ldots)$ be any point of J^{ω}. Choose some y_n then $| y_n | \leq 1/n$. For convenience, we will choose $n = 3$, $| y_3 | \leq 1/3$. Now choose r such that $| y_3 + r | < 1/3$ and consider the following family of points

$$\{x_n\} = (y_1, y_2, y_3 + r/n, y_4, \ldots)$$

where $n = 1, 2, 3 \ldots$, then certainly $\{x_n\} \in J^{\omega}$ and $\rho\,(x_n, y) < r$. Hence, every open sphere about y of radius r will contain points of J^{ω} other than y. Therefore, y is a limit point of J^{ω}. Since y was arbitrary, every point of J^{ω} is a limit point of J^{ω}. Thus, J^{ω} is a closed set and dense in itself, hence J^{ω} is a perfect set.

CHAPTER V

RELATIONSHIP OF HILBERT SPACE TO OTHER METRIC SPACES

Here we will be interested in developing various metric spaces and showing how they are related to E^{ω}. It will be shown that Euclidean n-space, denoted by E_n, is homeomorphic to a subspace of E^{ω}, and that the mapping between E_n and the subspace of E^{ω} is an isometry. Probably the most important feature of the chapter is that it will be proved, that every second countable regular Hausdorff Space is homeomorphic to a subset of J^{ω}.

<u>THEOREM 5.1</u>

Let W be the set of all sequences $\{x_n\}$ of real numbers such that for all n, $0 \leq x_n \leq 1$. Then, if $x = \{x_n\}$ and $y = \{y_n\}$, denote points of W, the mapping $f : W \times W \longrightarrow R$ defined by

$$f(x,y) = \sum_{n=1}^{\infty} 2^{-n} | x_n - y_n |$$

is a metric for W.

Proof:

Obviously, W is non-empty, since the point $0 = (0, 0, 0, \ldots)$ is in W . Now, for all points x and y in W , since $2^{-n} \mid x_n - y_n \mid \geq 0$ for all n , we have that $f(x,y) \geq 0$. Suppose that $f(x,y) = 0$, then

$$\sum_{n=1}^{\infty} 2^{-n} \mid x_n - y_n \mid = 0$$

however, this holds if and only if $x_n = y_n$ for all n , since $\mid x_n - y_n \mid \geq 0$ for all n . Conversely, if $x_n = y_n$ for all n , then $\mid x_n - y_n \mid = 0$ holds for all n , and consequently $f(x,y) = 0$. Consider, now for all $x,y \in W$,

$$f(x,y) = \sum_{n=1}^{\infty} 2^{-n} \mid x_n - y_n \mid = \sum_{n=1}^{\infty} 2^{-n} \mid y_n - x_n \mid = f(y,x)$$

also, for all $x,y,z \in W$

$$f(x,z) = \sum_{n=1}^{\infty} 2^{-n} \mid x_n - z_n \mid = \sum_{n=1}^{\infty} 2^{-n} \mid x_n - y_n + y_n - z_n \mid$$

$$\leq \sum_{n=1}^{\infty} 2^{-n} \left(\mid x_n - y_n \mid + \mid y_n - z_n \mid \right)$$

$$= \sum_{n=1}^{\infty} 2^{-n} \mid x_n - y_n \mid + \sum_{n=1}^{\infty} 2^{-n} \mid y_n - z_n \mid$$

$$= f(x,y) + f(y,z)$$

Therefore, $f(x,y)$ defines a metric for W .

THEOREM 5.2

The set W with the metric f is homeomorphic to J^{ω} .

Proof:

For $x = (x_1, x_2, x_3, \ldots)$ in J^{ω} , let F be a mapping of J^{ω} into W as follows

$$F(x_n) = \frac{1 - nx_n}{2} = w_n$$

where $w = (w_1, w_2, w_3, \ldots)$ is in W . Then what we have is a mapping of $x \longrightarrow w$. Now let $a = (a_1, a_2, a_3, \ldots)$ and $b = (b_1, b_2, b_3, \ldots)$ be two points J^{ω} such that $a \longrightarrow p$ and $b \longrightarrow p$ where $p \in W$. Then for each n ,

$$p_n = \frac{1 - na_n}{2} = \frac{1 - nb_n}{2}$$

$$a_n = b_n$$

Hence, $a = b$ and thus F is one-to-one. Now consider the following mapping

$$G(w_n) = \frac{1 - 2w_n}{n} = x_n$$

for $w = (w_1, w_2, w_3, \ldots)$ in W. This mapping carries $w \longrightarrow x$ and is likewise one-to-one. Thus, if we set $G(w_n) = F^{-1}(w_n)$ we will have that F is an onto mapping, since it is invertible.

We now wish to show that F is continuous. Given $\sigma > 0$, let $a = \{a_n\}$ be any point in J^{ω} with image $p = \{p_n\}$ in W. Choose

$$\delta = \frac{\sigma}{\sum_{n=1}^{\infty} n/2^{n+1}}$$

Then whenever $\rho\,(a,x) = \left[\sum_{n=1}^{\infty} (a_n - x_n)^2\right]^{1/2} < \delta$

we have that $f\,(p,w) = \sum_{n=1}^{\infty} 2^{-n} \mid p_n - w_n \mid$

$$= \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} \mid a_n - x_n \mid$$

Now, since $\sum_{n=1}^{\infty} (a_n - x_n)^2 < \delta^2$ we know that for each n, $(a_n - x_n)^2 < \delta^2$, hence, $| a_n - x_n | < \delta$. Therefore,

$$\sum_{n=1}^{\infty} \frac{n}{2^{n+1}} | a_n - x_n | < \sum_{n=1}^{\infty} \frac{n}{2^{n+1}} \frac{\sigma}{\sum_{n=1}^{\infty} n/2^{n+1}} = \sigma$$

Hence, F is continuous at $a \in J^{\omega}$ and since a was arbitrary, F is continuous at all points of J^{ω}.

Now we wish to show that F^{-1} is continuous. Let $p = \{p_n\}$ be any point of W with image $a = \{a_n\}$ in J^{ω}. Let $\sigma > 0$ be given. For each n, choose

$$\delta^2 = \frac{\sigma^2}{\sum_{n=1}^{\infty} \frac{2^{2n+2}}{n^2}}$$

then whenever $f(p,w) = \sum_{n=1}^{\infty} 2^{-n} | p_n - w_n | < \delta$

we have that $\rho(a,x) = \left[\sum_{n=1}^{\infty} (a_n - x_n)^2 \right]^{1/2}$

$$= \left[\sum_{n=1}^{\infty} 4n^{-2} (p_n - w_n)^2 \right]^{1/2} .$$

Now, since $\sum_{n=1}^{\infty} 2^{-n} \mid p_n - w_n \mid < \delta$ we have that for each n , $\mid p_n - w_n \mid < 2^n \delta$.

Thus, $(p_n - w_n)^2 < 2^{2n}\delta^2$ and

$$\left[\sum_{n=1}^{\infty} 4n^{-2} (p_n - w_n)^2 \right]^{1/2} < \left[\sum_{n=1}^{\infty} \frac{2^{2n+2}}{n^2} \frac{\sigma^2}{\sum_{n=1}^{\infty} \frac{2^{2n+2}}{n^2}} \right]^{1/2} = \sigma$$

Therefore, we have shown that F^{-1} is continuous at $p \in W$, and since p was arbitrary, F^{-1} is continuous at all points in W . Hence, W with the metric f is homeomorphic to J^{ω} .

<u>DEFINITION 5.1</u> If (X,ρ) and (Y,σ) are metric spaces, and if f is a mapping of X onto Y , then <u>f is an isometry</u> if and only if

$$\rho\,(x,y) = \sigma\,(f(x), f(y)) .$$

<u>THEOREM 5.3</u>

Each Euclidean n-space can be embedded in E^{ω} .

Proof:

Let $x = (x_1, x_2, \ldots , x_n)$ be some point of E_n .

Let f be a mapping of E_n into E^ω , which carries the point x of E_n into the point

$$x' = (x_1, x_2, \ldots, x_n, 0, 0, \ldots)$$

Let

$$S = \{x' = (x_1, x_2, \ldots, x_n, 0, 0, \ldots) \mid x_m = 0 \text{ for all } m > n\}$$

then, f defines a mapping of E_n onto S , which is a subset of E^ω . Let σ be the distance function on E_n , and for $x, y \in E_n$, let

$$\sigma(x,y) = \Big[\sum_{m=1}^{n} (x_m - y_m)^2\Big]^{1/2}$$

Now consider

$$\rho(f(x), f(y)) = \rho(x', y') = \Big[\sum_{m=1}^{\infty} (x_m - y_m)^2\Big]^{1/2}$$

$$= \Big[\sum_{m=1}^{n} (x_m - y_m)^2 + \sum_{m=n+1}^{\infty} (x_m - y_m)^2\Big]^{1/2}$$

$$= \Big[\sum_{m=1}^{n} (x_m - y_m)^2\Big]^{1/2} = \sigma(x,y)$$

since for all $m > n$, $x_m = y_m = 0$. Thus, f is an isometry between E_n and S .

DEFINITION 5.2 A base for the neighborhood system of a point x , or a local base at x , is a family of neighborhoods of x such that every neighborhood of x contains a member of the family.

DEFINITION 5.3 A topological space X , which has a base β , which is a countable family, is said to satisfy the second axiom of countability.

DEFINITION 5.4 A function f on a topological space X to another topological space Y is open, if and only if, the image of each open set is open.

DEFINITION 5.5 A topological space is said to be completely normal, if and only if, every one of its subspaces is normal.

LEMMA 5.1

Every regular space with a countable basis is completely normal.

(for proof see appendix)

Now that we have assembled these definitions and lemmas, we are in a position to assault the most important result of this chapter, which is stated as the next theorem.

THEOREM 5.4*

Every second countable regular Hausdorff Space X is homeomorphic to a subset of the Hilbert Cube.

Proof:

Let $\beta = \{B_i \mid i = 1, 2, 3, \ldots\}$ be the countable basis of X. By lemma 5.1, X is normal, and from the regularity of X, there are pairs of elements of β such that $\overline{B}_i \subset B_j$. Since β is countable, the collection of all such pairs is again countable; let us call it

$$P = \{P_n \mid n = 1, 2, \ldots\}$$

where $P_n = (B_i^n, B_j^n)$ and $\overline{B_i^n} \subset B_j^n$. Now clearly $\overline{B_i^n}$ and $X - B_j^n$ are closed and $\overline{B_i^n} \cap (X - B_j^n) = \phi$, hence by lemma 2.1, a mapping

$f_n : X \longrightarrow [0,1]$ can be defined

such that $f_n(\overline{B_i^n}) = 0$ and $f_n(X - B_j^n) = 1$. Finally, define $f : X \longrightarrow J^\omega$ by

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*John D. Baum, Elements of Point Set Topology. Englewood Cliffs, New Jersey: Prentice Hall Inc., 1964. Theorem No. 5.1.3, pp. 129-130.

$$f(x) = \left[\frac{f_n(x)}{n} \,\middle|\, n = 1, 2, 3, \ldots \right] ,$$

since for each x , $0 \leq f_n(x) \leq 1$, $f(x) \in J^\omega$.

First we wish to show that f is one-to-one. Let $x \neq y$, since X is Hausdorff, there exists open sets, which we may choose as basic sets B, B' such that $x \in B$ and $y \in B'$ and $B \cap B' = \phi$. Further, since X is normal, there exists $B'' \in \beta$ such that $x \in B'' \subset \overline{B}'' \subset B$, then $x \in \overline{B}''$ and $y \in X - B$ and the pair $(B'',B) \in P$, that is, for some n , $(B'',B) = (B_i^n , B_j^n)$. Thus,

$$f_n(x) = f_n(\overline{B_i^n}) = f_n(\overline{B''}) = 0$$

$$f_n(y) = f_n(X - B_j^n) = f_n(X - B) = 1$$

therefore, $f(x) \neq f(y)$, since $f(x)$ differs from $f(y)$ at the n-th place.

Now we show that f is continuous. Let $x \in X$ and let $\sigma > 0$. We wish to construct $U \in U_x$, the neighborhood system of $x \in X$, such that for any $y \in U$

$$\rho\ (f(x),f(y)) < \sigma \quad \text{in} \quad J^\omega$$

First, since for any point $y \in X$, $0 \leq f_n(y) \leq 1$, we have that

$$| f_n(x) - f_n(y) |^2 \leq 1$$

The infinite series $\sum_{n=1}^{\infty} 1/n^2$ converges, so that for sufficiently large N ,

$$\sum_{n=1}^{\infty} 1/n^2 < \sigma^2/2$$

whence

$$\sum_{n=N}^{\infty} | f_n(x) - f_n(y) |^2 (1/n^2) < \sum_{n=N}^{\infty} 1/n^2 < \sigma^2/2$$

Now let $k < N$, then the function $f_k : X \longrightarrow [0,1]$ is continuous, thus there is a $U_k \in U_x$ such that for $y \in U_k$

$$| f_k(x) - f_k(y) | < \frac{k\sigma}{[2(N-1)]^{1/2}}$$

$$\frac{| f_k(x) - f_k(y) |^2}{k^2} < \frac{\sigma^2}{2(n-1)}$$

Now let $U = \bigcup_{k=1}^{N-1} U_k$, then for $y \in U$

$$\sum_{n=1}^{\infty} \frac{|\ f_n(x) - f_n(y)\ |^2}{n^2} = \sum_{n=1}^{N-1} \frac{|\ f_n(x) - f_n(y)\ |^2}{n^2}$$

$$+ \sum_{n=N}^{\infty} \frac{|\ f_n(x) - f_n(y)\ |^2}{n^2}$$

$$< \frac{(N-1)\ \sigma^2}{2(N-1)} + \frac{\sigma^2}{2} = \sigma^2$$

Hence, $\rho\ (f(x),f(y)) < \sigma$ which implies that f is continuous.

Finally, we must show that f is an open mapping. Let U be open in X and let $x \in U$, then there exists $B_i, B_j \in \beta$ such that

$$x \in B_i \subset \overline{B}_i \subset B_j \subset U$$

by the normality of X and the fact that β is a basis. Thus, the pair (B_i, B_j) belonging to P, say

$(B_i, B_j) = (B_i^n\ , B_j^n)$. Then

$$f_n(x) = f_n(\overline{B_i^n}) = 0$$

and since $X - U \subset X - B_j^n$

$$f_n(X - U) = f_n(X - B_j^n) = 1$$

so that for $y \in X - U$

$$\rho\ (f(x),f(y)) = \left[\sum_{n=1}^{\infty} \frac{|\ f_n(x) - f_n(y)\ |^2}{n^2}\right]^{1/2}$$

$$> \left[\frac{f_n(x) - f_n(y)\ |^2}{n^2}\right]^{1/2} = \frac{1}{n}$$

Let $V = S\ (f(x),1/n)$ be any open sphere about $f(x)$ of radius $1/n$. If $z \in V$ then $\rho\ (f(x), z) < 1/n$ and $f^{-1}(z) \in U$, for if not, then $f^{-1}(z) \in X - U$ and $\rho\ (f(x), f(f^{-1}(z))) > 1/n$, a contradiction. Thus $f^{-1}(V) \subset U$ and $x \in V \subset f(U)$. Whence, $f(U)$ is open since it contains a neighborhood of each of its points. Consequently, we have proved that f is a one-to-one continuous open mapping, hence f is a homeomorphism.

<u>DEFINITION 5.6</u> The <u>diameter of a subset A</u> of a metric space (X,ρ) is

$$\sup \{\rho\ (x,y)\ |\ x \in A \text{ and } y \in A\}$$

Using this definition and some intuitive reasoning, we would now like to determine the diameter of the Hilbert Cube.

<u>THEOREM 5.5</u>

The diameter of J^{ω} is $\frac{\pi\sqrt{6}}{3}$.

Proof:

Consider the following two sequences $x = \{1/n\}$ and $y = \{-1/n\}$. Clearly, x and y are members of J^{ω}. Now, for each n, these points of J^{ω} represent sequences that are at maximum distance from each other. Hence, x and y are at maximum distance from each other in J^{ω} and the distance between them will be the diameter of J^{ω}.

$$\rho\,(x,y) = \left[\sum_{n=1}^{\infty} (1/n - (-1/n))^2\right]^{1/2} = \left[\sum_{n=1}^{\infty} 4/n^2\right]^{1/2}$$

If one now considers the Fourier Series for $f(x) = x^2$ on $[-\pi,\pi]$ we get

$$x^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n \cos(nx)}{n^2}$$

let $x = \pi$, then

$$\pi^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} 1/n^2$$

$$\frac{2\pi^2}{3} = 4\sum_{n=1}^{\infty} 1/n^2$$

Therefore, $\rho\,(x,y) = \frac{\pi\sqrt{6}}{3}$.

APPENDIX

The following is a list of lemmas and theorems that appear in this thesis without formal proof. Below each such lemma or theorem is the location of the proof, if the reader desires such proof. Since all books appear in the List of References, only the name of the author, theorem number, and page are given below.

1. Lemma 1.1
   Kelley, Theorem 4.8, pg. 120.
2. Lemma 2.1
   Baum, Lemma 5.12, pg. 127.
3. Lemma 2.2
   Kelley, Theorem 4.5, pg. 116.
4. Theorem 2.7
   Kelley, Theorem 4.7, pg. 118.
5. Lemma 3.1
   Hall and Spencer, Theorem 14.6, pg. 109.
6. Lemma 4.4
   Kelley, Theorem 1.21, pg. 54.
7. Theorem 4.4
   Hall and Spencer, Theorem 7.8, pg. 44.
8. Lemma 5.1
   Baum, Theorem 3.24, pg. 88.

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