1967

Measure theory

James William Cook
Southern Illinois University Edwardsville

MEASURE THEORY

A Thesis

Submitted to the Graduate Faculty of

Southern Illinois University

Edwardsville, Illinois

in Partial Fulfillment of the

Requirements for the Degree of

Master of Arts

in

The Department of Mathematics

by

James William Cook

A. B. Washington University

Saint Louis, Missouri, 1962

June, 1967

CONTENTS

DEDICATION

To my wife, Wilma , who provided the time and place to produce this paper, the moral support when the ideas didn't come, the understanding when the nerves were frayed, but most of all for the feeling of being loved even if unable to produce artistically, this paper is respectfully dedicated.

ACKNOWLEDGMENT

Grateful acknowledgment is due to Professor Andrew Lindstrum for his many helpful suggestions without which this paper could not have been completed, for his time in reading the many very rough drafts and for demanding that they be made rigorous, but thanks most of all for the academic freedom to explore without restriction as to content or style and write an intellectually honest paper.

INTRODUCTION

Since Lebesgue measure is complete[22], every subset of a set of Lebesgue measure zero is measurable. So if a set,S, is non-measurable then no set of Lebesgue measure zero contains S. Now every set of Lebesgue measure greater than zero contains a non-measurable subset[23]. Hence a necessary condition for non-measurability of a set S is that S be contained in a set of Lebesgue measure greater than zero and not be contained in any set of Lebesgue measure zero.

There exists a Cantor set of Lebesgue measure greater than zero[24] implies this set contains a non-measurable subset P. Now all Cantor sets are homeomorphic with the Cantor discontinuum of Lebesgue measure zero implies P has a homeomorphic image P' contained in the Cantor discontinuum. Now P' is measurable means we can find two homeomorphic sets, one measurable and the other non-measurable. Hence, we can not determine the measurability or non-measurability of a set by its topological properties alone.

The purpose of this paper is to construct a measure of real Euclidean n-space that is equivalent to Lebesgue measure for all the "usual" sets, but can determine the measurability of a set by its topological properties.

Since any metric geometry in a Euclidean space requires the axiom of distance[24], our measure must preserve the lengths of intervals. This means our measure must be non-trivial (trivial measure: S is measurable implies m(S) = 0) . By the axiom of free mobility[25], our measure must be translation invariant. If we do not have countable additivity, we cannot consider the limits of sequences of sets and our measure would have the limitations of the Jordan content theory. This leaves only completeness to be considered.

The set given by Halmos[23] is non-measurable with respect to any non-trivial, translation invariant, countably additive measure, and it is formed by taking the intersection of an everywhere dense non-measurable set with a set of measure greater than zero. Hence, in our measure the intersection of a Cantor set of measure greater than zero with this everywhere dense non-measurable set must be non-measurable. Now this subset of a Cantor set of Lebesgue measure greater than zero has, as a homeomorphic image, the corresponding subset of the Cantor set of measure equal to zero (since all Cantor sets are homeomorphic). If the measurability of a set is to be determined by its topological properties, then the homeomorphic image of a non-measurable set must be non-measurable. Hence, there exists a non-measurable subset of a set of measure zero in any non-trivial, translation invariant, countably additive measure whose definition of measurability is entirely in terms of topological properties.

Hence, this paper will present a non-complete "Lebesgue" measure of Euclidean n-space.

CHAPTER I

Preliminaries

In the following 38 pages all sets are bounded subsets of the set, R, of real numbers.

Definition 1 A set T is an interval iff 1) $\exists\, a,b \in R \ni a < b$ and $a < x < b \Rightarrow x \in T$ and 2) $x > b$ or $x < a \Rightarrow x \notin T$.

Clearly a or b or both may or may not be elements of T. T is a closed interval iff both $a,b \in T$ and T is an open interval iff both $a,b \notin T$.

Definition 2 A set S is a neighborhood of a point x iff $\exists$ an open interval $I \ni x \in I \subset S$.

Definition 3 A point x is a point of condensation of a set S iff every neighborhood of x contains a non-enumerable number of points of S.

Definition 4 $E_s = \{x : x \in S,$ x is a point of condensation of S$\}$.

Since any bounded non-enumerable set, S, has at least one point of condensation which is an element of S^1, then $E_s = \emptyset \Rightarrow$ S is at most enumerable. S is at most enumerable $\Rightarrow$ S has no points of condensation. This proves $E_s = \emptyset$ iff S is at most enumerable and motivates the following definition.

Definition 5 If $E_S = \emptyset$ then S is measurable, ⓜ, and $m(S) = 0$.

Since $S-E_S$ is at most enumerable[2], E_S contains all but an at most enumerable number of S. Then E_S and S differ by an at most enumerable set, which is of measure zero, and we shall define the measure of S by the measure of E_S.

Definition 6 $M = [\text{g.l.b.}(E_S), \text{l.u.b.}(E_S)]$, a closed interval.

The following definition is made to keep all sets under discussion bounded.

Definition 7 If $E_S \neq \emptyset$, then C(S) is the complement of S in M.

Definition 8 The usual topology for the real numbers is the family of all those sets, S, $\ni \forall x \in S \ \exists$ an open interval $I \subset S \ni x \in I$[3].

Definition 9 If (S,U) is a topological space, where S is a set of real numbers, U is the topology on S and $\emptyset \neq A \subset S$, then $\{u \cap A : u \in U\}$ is the relative topology of A[4].

It follows from definitions 8 and 9 that, if (S,U) is a topological space, U is the usual topology, and A is a set $\ni \emptyset \neq A \subset S$ then $\{u \cap A : u \in U\}$ is the <u>relative usual topology</u> of A.

<u>Theorem 1</u> If T is an interval contained in E_S, then $\exists$ a maximal interval $I_0 \ni T \subset I_0 \subset E_S$, i.e: I_1 is an interval and I_1 properly contains $I_0 \Rightarrow I_1 \not\subset E_S$.

<u>Proof</u>: Let $I = \bigcup_{\alpha}\{I_\alpha : I_\alpha \text{ is an interval}, I_\alpha \subset E_S\}$ with the relative usual topology of E_S. T is an interval $\Rightarrow$ T is connected[5] and $\exists$ a maximal connected set I_0 which contains T.[6] Since I_0 has at least two points it is an interval[5] and we can define the set $\mathcal{S} \subset I$ of maximal disjoint intervals, each of which is contained in E_S[6].

<u>Corollary 1</u> If T is an interval contained in M $\ni$ 1) $T \cap E_S \neq \emptyset$ and 2) $\forall$ interval $I \subset T$, $I \not\subset E_S$, then $\exists$ a maximal closed interval $K_0 \ni$ 1) $T \subset K_0 \subset M$ and 2) $\forall K_1 \subset M$ K_1 properly contains $K_0 \Rightarrow \exists$ an interval $I_1 \subset K_1 \ni I_1 \subset E_S$.

<u>Proof</u>: Let $K = \bigcup_{\gamma}\{K_\gamma : K_\gamma \text{ is an interval}, K_\gamma \cap E_S \neq \emptyset,$ and $\forall$ interval $I \subset K_\gamma$, $I \not\subset E_S\}$ with the relative usual topology and the proof is clear from theorem 1. We can define the set $\mathcal{K} \subset K$ of maximal disjoint intervals[6]

Now $K_1 \in \mathcal{K} \Rightarrow K_1$ is closed.(else $\exists K_2 = K_1 \cup \{$ the endpoints of $K_1\}$ which properly contains $K_1 \ni K_2$ has no subinterval contained in E_S) The endpoints of K_1 are elements of the closure of E_S.(else K_1 is not maximal)

Corollary 2 If T is an interval $\ni T \subset C(E_S)$ and $\exists K_0 \in \mathcal{K} \ni T \subset K_0$ then $\exists$ a maximal interval $J_0 \ni$ $T \subset J_0 \subset K_0$ and $J_0 \subset C(E_S)$

Proof: Let $J = \bigcup_{\varrho}\{ J_\varrho : J_\varrho$ is an interval $\subset C(E_S)$, $\exists K_\varrho \in \mathcal{K} \ni J_\varrho \subset K_\varrho\}$ with the relative usual topology and the proof of theorem 1 $\Rightarrow$ T $\subset$ in a component which is an interval. Again we can define the set $\mathcal{J} \subset J$ of maximal disjoint intervals, each of which is contained in $C(E_S)$, i.e: $J_1 \in \mathcal{J}$ and J_2 properly contains $J_1 \Rightarrow J_2 \not\subset C(E_S)$, and is contained in some $K \in \mathcal{K}$[6].

Lemma 1 $\nexists$ an enumerable set of E_S in an open interval which is otherwise contained in $C(E_S)$.

Proof: If $x \in E_S$ and $x \in$ an open interval, U, which contains only an enumerable number of E_S, then (since $S-E_S$ is enumerable and $U \cap E_S$ is enumerable $\Rightarrow$ $[(S-E_S)\cap U] \cup (U \cap E_S) = U \cap S$ is enumerable[7]) $\exists$ a neighborhood U of x that does not contain a non-enumerable number of S $\Rightarrow x \notin E_S$, which is false.

Theorem 2 For every set S, $\{\bigcup \mathcal{I}\} \cup \{\bigcup \mathcal{K}\} \supset E_S$

Proof: If $E_S = \emptyset$ the theorem is trivially true. If $E_S \neq \emptyset$ then, by lemma 1, $\forall x \in E_S$ and $\forall$ neighborhood U of x, U contains a non-enumerable number of E_S. Hence, if $x \notin$ some $I \in \mathcal{I}$ then $x \in$ some K

Corollary For every set S, $\{\cup\mathcal{I}\}\cup\{\cup\mathcal{K}\}-\{\cup\mathcal{J}\}\supset E_s$

Proof: Since $x\in E_S \Rightarrow x\notin\{\cup\mathcal{J}\}$, then $C(\{\cup\mathcal{J}\})\supset E_S$. Now, from theorem 2, $\{\cup\mathcal{I}\}\cup\{\cup\mathcal{K}\}\supset E_S$. Therefore,

$[\{\cup\mathcal{I}\}\cup\{\cup\mathcal{K}\}]\cap C(\{\cup\mathcal{J}\})\supset E_S$ and hence,

$\{\cup\mathcal{I}\}\cup\{\cup\mathcal{K}\}-\{\cup\mathcal{J}\}\supset E_S$.

Theorem 1 and its corollaries justify definitions 10, 11, and 12.

Definition 10 Let $\mathcal{I}_S = \{I_i : i\in\alpha\}$ be the set of maximal disjoint intervals, each of which is contained in E_S, i.e: $I_1\in\mathcal{I}_S$ and I_2 properly contains $I_1 \Rightarrow I_2\not\subset E_S$.

Definition 11 Let $\mathcal{K}_S = \{K_\kappa : \kappa\in\gamma\}$ be the set of maximal disjoint closed intervals, each of which contains points of E_S, but has no subinterval contained in E_S (i.e. $K_1\in\mathcal{K}_S$ and K_2 properly contains $K_1 \Rightarrow K_2$ has a subinterval $\subset E_S$) and is contained in M (i.e. $K\in\mathcal{K}_S \Rightarrow K\subset[\text{g.l.b.}(E_S), \text{l.u.b.}(E_S)]$).

Definition 12 Let $\mathcal{J}_S = \{J_j : j\in\beta\}$ be the set of maximal disjoint intervals, each of which is contained in $C(E_S)$ (i.e. $J_1\in\mathcal{J}_S$ and J_2 properly contains $J_1 \Rightarrow J_2\not\subset C(E_S)$) and is contained in some $K\in\mathcal{K}_S$ (i.e. $J\in\mathcal{J}_S \Rightarrow \exists\, K\in\mathcal{K}_S \ni K\supset J$).

Lemma 2 The number of elements of $\mathcal{I}_s$, $\mathcal{J}_s$, and $\mathcal{K}_s$ is each at most enumerable.

Proof: Each element of $\mathcal{I}_s$ is an interval and every interval on the real line contains a rational number. Since these intervals are disjoint each of them contains a rational number not contained in any of the others. Hence the elements of $\mathcal{I}_s$ can be put into 1 - 1 correspondence with a subset of the rational numbers and so the number of elements of $\mathcal{I}_s$ is at most enumerable. Similarly, the number of elements of $\mathcal{J}_s$ and $\mathcal{K}_s$ are each at most enumerable.

CHAPTER II

Case I Sets

<u>Definition 13</u> A <u>set</u> S <u>is</u> <u>of</u> <u>case</u> I iff M can be partitioned into a set of disjoint intervals, $\{A_i\}$, $\ni$ $\forall A_n \in \{A_i\}$ either the number of components of E_s in A_n or the number of components of $C(E_s)$ in A_n is at most enumerable.

<u>Theorem 3</u> If a set is of case I, then the sets E_s and $[\{\cup\mathcal{I}_s\}\cup\{\cup\mathcal{K}_s\} - \{\cup\mathcal{J}_s\}]$ differ by an at most enumerable set of points.

<u>Proof</u>: Let W = $\{\cup\mathcal{I}_s\}\cup\{\cup\mathcal{K}_s\} - \{\cup\mathcal{J}_s\}$

By the corollary to theorem 2, $W \supset E_s$. So, if W and E_s differ by a non-enumerable set of points, then $W-E_s$ contains a non-enumerable subset of $C(E_s)$. Now $x \in W \Rightarrow$ $x \notin$ any $I \in \mathcal{I}_s$ and $x \notin$ any $J \in \mathcal{J}_s \Rightarrow \exists K_m \in \mathcal{K}_s \ni$ $x \in K_m$. By lemma 2 the number of elements of $\mathcal{K}_s$ is enumerable $\Rightarrow \exists K_n \in \mathcal{K}_s \ni K_n$ contains a non-enumerable set of $C(E_s)\cap W$, and since the number of $\{A_i\}$, (as described in definition 13), is at most enumerable and $\bigcup_i\{A_i\} \supset M$ then $\exists A_n \in \{A_i\} \ni A_n \cap K_n \cap W$ $= A_n \cap K_n \cap C(E_s)$ is non-enumerable.

$x \in A_n \cap K_n \cap W \Rightarrow \{x\}$ is a component of $C(E_s)$ (since $x \notin$ any $J \in \mathcal{J}_s$)[5]. Hence, $A_n \cap K_n$ contains a non-enumerable number of components of $C(E_s)$. A_n is an interval, K_n is an interval, and $A_n \cap K_n$ contains at least two points $\Rightarrow A_n \cap K_n$ is an interval. The interior, $(A_n \cap K_n)^\circ$, of $A_n \cap K_n$ contains points of E_s (else $(A_n \cap K_n)^\circ \subset C(E_s) \Rightarrow (A_n \cap K_n)^\circ$ contains at most one component of $C(E_s)$, namely $(A_n \cap K_n)^\circ$). By definition 3, $x \in (A_n \cap K_n)^\circ \cap E_s \Rightarrow$ every neighborhood of x contains a non-enumerable number of E_s[1]. By definition 11 no subinterval of K_n is contained in E_s, hence $x \in E_s \cap K_n \cap A_n \Rightarrow \{x\}$ is a component of E_s[5]. Therefore, $A_n \cap K_n$ and hence A_n contains a non-enumerable number of components of both E_s and $C(E_s) \Rightarrow S$ is not of case I.

<u>Definition 14</u> $\overline{E}_s$ is the <u>closure</u> of E_s.

<u>Definition 15</u> Let $E_s \neq \emptyset$. A case I set, S, is <u>measurable</u>, ⓜ, iff $\exists$ a monotone function, f, on a closed interval $[a,b] \subset R \ni f([a,b]) = \overline{E}_s$. Any such f is a <u>measure function</u> for S.

If a case I set, S, is ⓜ and $E_s \neq \emptyset$, then, by definition 15, $\exists$ a monotone function, f, on a closed interval $[a,b] \subset R \ni f([a,b]) = \overline{E}_s$. Since $I_1 \in \mathcal{J}_s \Rightarrow I_1 \subset E_s$ then the closure of $I_1 \subset \overline{E}_s$ and the endpoints of the closure of I_1 are elements of $\overline{E}_s$. Hence $\forall I_i \in \mathcal{J}_s$ $\exists x_i, y_i \in [a,b] \ni \{f(x_i), f(y_i)\} = \{\text{l.u.b.}(I_i), \text{g.l.b.}(I_i)\}$ Every $K_1 \in \mathcal{K}_s$ is a closed interval whose endpoints are elements of $\overline{E}_s$ (else $\exists K_2 = K_1 \cup \{$the endpoints of $K_1\}$ that properly contains $K_1 \ni K_2$ has no subinterval $\subset E_s$). Hence $\forall K_k \in \mathcal{K}_s$ $\exists x_k, y_k \in [a,b] \ni \{f(x_k), f(y_k)\} = \{\text{l.u.b.}(K_k), \text{g.l.b.}(K_k)\}$. Every $J_1 \in \mathcal{G}_s$ has a l.u.b.$(J\ ) \in \overline{E}_s$ and g.l.b.$(J_1\ ) \in \overline{E}_s$ (since every neighborhood of l.u. b.(J_1) or g.l.b.(J_1) contains a non-enumerable number of E_S, else l.u.b.(J_1) or g.l.b.(J_1) is an interior point of J_1 $\Rightarrow$ J_1 is not maximal in C(E_S)). Hence $\forall J_j \in \mathcal{J}_S$ $\exists x_j, y_j \in [a,b] \ni$ $\{f(x_j), f(y_j)\} = \{$ g.l.b.(J_j), l.u.b.(J_j) $\}$.

This discussion and theorem 3 motivates the following definition of the measure of a ⓜ case I set with respect to a measure function f.

Definition 16 If a case I set, S, is ⓜ with a measure function f on [a,b], then we define

$$m_f(S) = \left| \sum_{i\in\alpha}(f(x_i) - f(y_i)) + \sum_{k\in\nu}(f(x_k) - f(y_k)) - \sum_{j\in\beta}(f(x_j) - f(y_j)) \right|$$

where α, ν, and β are indexing sets on $\mathcal{I}_S$, $\mathcal{K}_S$, and $\mathcal{J}_S$ respectively, and $x_i > y_i$, $x_k > y_k$, and $x_j > y_j$, where all $x,y \in [a,b]$. By lemma 2 the set of natural numbers is an adequate indexing set for α, ν, or β.

To shorten the notation, let

$$m_f(S) = \left| \sum_{i\in\alpha} F_i + \sum_{k\in\nu} F_k - \sum_{j\in\beta} F_j \right|$$

Lemma 3 $\forall$ bounded interval S, S is a ⓜ case I set and $\forall$ measure function f, $m_f(S)$ = l.u.b.(S) − g.l.b.(S)

Proof: Since S is a non-empty bounded set, it has greatest lower and least upper bounds. Then $\exists$ an interval [g.l.b.(S), l.u.b.(S)] $\subset$ R and a monotone function

$f(x) = x \ni f([\text{g.l.b.}(S), \text{l.u.b.}(S)]) =$
$[\text{g.l.b.}(S), \text{l.u.b.}(S)] = \overline{E}_s$. Now any monotone function $f \ni f([c,d]) = \overline{E}_s \Rightarrow \{f(c), f(d)\} = \{\text{l.u.b.}(S), \text{g.l.b.}(S)\}$, and $E_s = S \Rightarrow \mathcal{Q}_s = \{S\}$, $\mathcal{J}_s = \emptyset$, $\mathcal{K}_s = \emptyset$. Hence, $m_f(S) = |f(c) - f(d)| = |f(d) - f(c)| = \text{l.u.b.}(S) - \text{g.l.b.}(S)$.

<u>Lemma 4</u> If I_1 and I_2 are bounded intervals and $I_1 \supset I_2$ then $m_f(I_1) \geq m_f(I_2)$.

<u>Proof</u>: By lemma 3

<u>Definition 17</u> A function $f([a,b]) = \overline{E}_s$ is <u>isotone</u> iff $x_i, y_i \in [a,b]$ and $x_i \geq y_i \Rightarrow f(x_i) \geq f(y_i)$ and a function is <u>antitone</u> iff $x_i, y_i \in [a,b]$ and $x_i \geq y_i \Rightarrow f(x_i) \leq f(y_i)$[8].

<u>Lemma 5</u> If S is a Ⓜ case I set then $\left|\sum_{i\in\alpha} F_i + \sum_{k\in\gamma} F_k - \sum_{j\in\beta} F_j\right| = \sum_{i\in\alpha}|F_i| + \sum_{k\in\gamma}|F_k| - \sum_{j\in\beta}|F_j|$.

<u>Proof</u>: Since f is monotone then, <u>if</u> f <u>is</u> <u>isotone</u> $x_i > y_i \Rightarrow f(x_i) > f(y_i)$ (since $f(x_i)$ and $f(y_i)$ are the endpoints of a non-degenerate interval). Hence, $f(x_i) - f(y_i) = F_i > 0 \Rightarrow |F_i| = F_i$. Similarly, $x_k > y_k \Rightarrow |F_k| = F_k$ and $x_j > y_j \Rightarrow |F_j| = F_j$. By definition 12 $\forall J \in \mathcal{J}_s \; \exists K \in \mathcal{K}_s \ni K \supset J$ and by lemma 4, this $\Rightarrow m_f(K) \geq m_f(J) \Rightarrow \sum F_k - \sum F_j \geq 0 \Rightarrow \sum F_i + \sum F_k - \sum F_j \geq 0$. Hence $\sum_{i\in\alpha}|F_i| + \sum_{k\in\gamma}|F_k| - \sum_{j\in\beta}|F_j| =$

$\sum_{i\in\alpha} F_i + \sum_{k\in\gamma} F_k - \sum_{j\in\beta} F_j = \left| \sum_{i\in\alpha} F_i + \sum_{k\in\gamma} F_k - \sum_{j\in\beta} F_j \right|$

<u>If</u> f <u>is</u> <u>antitone</u>, then $x_i > y_i \Rightarrow f(x_i) < f(y_i) \Rightarrow$ $f(x_i) - f(y_i) = F_i < 0 \Rightarrow |F_i| = -F_i$ and similarly $|F_k| = -F_k$, $|F_j| = -F_j$. Again, by lemma 4, $\sum |F_k| - \sum |F_j| > 0$ and hence $\sum |F_i| + \sum |F_k| - \sum |F_j| > 0$. Then $\sum |F_i| + \sum |F_k| - \sum |F_j| = \left| \sum |F_i| + \sum |F_k| - \sum |F_j| \right| = \left| \sum -F_i + \sum -F_k - \sum -F_j \right| = \left| -[\sum F_i + \sum F_k - \sum F_j] \right| = \left| \sum F_i + \sum F_k - \sum F_j \right|$.

<u>The</u> <u>measure</u> <u>of</u> <u>a</u> <u>case</u> I <u>set</u> <u>is</u> <u>unique</u>.

<u>Theorem 4</u> If f and g are measure functions for a case I set, S, then $m_f(S) = m_g(S)$

<u>Proof</u>: Since each $I_i \in \mathcal{I}_S$ is a maximal bounded set, then the g.l.b.(I_i) and l.u.b.(I_i) are unique. Then f and g are measure functions for S $\Rightarrow \forall\, I_i \in \mathcal{I}_S\ \exists\, x_i, y_i, u_i, v$ $\ni f(x_i), f(y_i), g(u_i), g(v_i) \in \overline{E}_S$ and $|$l.u.b.(I_i) $-$ g.l.b.$(I_i)| = |f(x_i) - f(y_i)| = |g(u_i) - g(v_i)|$ Similarly, $\forall\, K_k \in \mathcal{K}_S\ \exists\, x_k, y_k, u_k, v_k \ni f(x_k), f(y_k)$, $g(u_k), g(v_k) \in \overline{E}_S$ and $|$l.u.b.$(K_k) -$ g.l.b.$(K_k)| =$ $|f(x_k) - f(y_k)| = |g(u_k) - g(v_k)|$ and $\forall\, J_j \in \mathcal{J}_S$ $\exists\, x_j, y_j, u_j, v_j \ni f(x_j), f(y_j), g(u_j), g(v_j) \in \overline{E}_S$ and $|$l.u.b.$(J_j) -$ g.l.b.$(J_j)| = |f(x_j) - f(y_j)| = |g(u_j) - g(v_j)|$. Now, by lemma 5, $m_f(S) = \left| \sum F_i + \sum F_k - \sum F_j \right| = \sum |F_i| + \sum |F_k| - \sum |F_j| = \sum |G_i| + \sum |G_k| - \sum |G_j| = \left| \sum G_i + \sum G_k - \sum G_j \right| = m_g(S)$

Definition 18 Set T is the image under a translation of a set S iff $\exists$ a unique $c \in R \ni T = \{t : t = y + c, y \in S\}$

The measure of a case I set is invariant under a translation.

Theorem 5 If S is a Ⓜ case I set and $\exists c \in R \ni$ $T = \{t : t = y + c, y \in S\}$, then T is Ⓜ and $m(S) = m(T)$

Proof: Any translation on R with the usual metric, ($\rho(x,y) = |x - y|$), preserves distance, i.e: $|y_1 + c - (y_2 + c)| = |y_1 - y_2|$)

1) $\forall y_0 \in S$ $\exists$ a 1-1 correspondence between the neighborhoods U_s of y_0 and the neighborhoods U_t of $y_0 + c$ ($U_s \leftrightarrow U_t = \{t : t = y + c, y \in U_s\}$). Therefore, $U_s \supset$ a non-enumerable number of S iff $U_t \supset$ a non-enumerable number of T.

Then $E_s = \emptyset$ iff $E_t = \emptyset$ and hence either $\Rightarrow m(S) = 0 = m(T)$ So $E_s \neq \emptyset$ iff $E_t \neq \emptyset$ and $y_1 \in E_s$ iff $y_1 + c \in E_t$. Now g is a measure function for $S \Rightarrow f(x) = g(x) + c$ is a measure function for $T \Rightarrow$ T is Ⓜ and $\exists [a,b] \subset R \ni$ $\forall x \in [a,b]$, $f(x) = t = y + c$ where $y \in \overline{E}_s$. Since the g.l.b. and the l.u.b. of a bounded interval are unique, if $y_1 + c$ is a l.u.b.(or g.l.b.) for $I_i \in \mathcal{S}_t$, then by 1) $\exists r_i \in \mathcal{S}_s \ni y_1$ is a l.u.b.(or g.l.b.) for $r_i = \{y : y \in S, y + c \in I_i\}$ and conversely y is a l.u.b.(or g.l.b.) for $r_i \Rightarrow y_1 + c$ is a l.u.b.(or g.l.b.) for $r_i \Rightarrow y + c$ is a l.u.b.(or g.l.b.) for

$I_i = \{t : t - c \in r_i\}$. Hence $\exists$ the 1-1 correspondence $I_i \leftrightarrow r_i$ and $\alpha_s \leftrightarrow \alpha_t$. Similarly $\mathcal{K}_s \leftrightarrow \mathcal{K}_t \Rightarrow$ $\nu_s \leftrightarrow \nu_t$ and $\mathcal{J}_s \leftrightarrow \mathcal{J}_t \Rightarrow \varrho_s \leftrightarrow \varrho_t$. Then

$$m(S) = \left| \sum_{i\in\alpha}(g(x_i) - g(y_i)) + \sum_{K\in\nu}(g(x_k) - g(y_k)) - \sum_{j\in\varrho}(g(x_j) - g(y_j)) \right| = \left| \sum_{i\in\alpha}(g(x_i) + c - [g(y_i) + c]) + \sum_{K\in\nu}(g(x_k) + c - [g(y_k) + c]) - \sum_{j\in\varrho}(g(x_j) + c - [g(y_j) + c]) \right| = \left| \sum_{i\in\alpha}(f(x_i) - f(y_i)) + \sum_{K\in\nu}(f(x_k) - f(y_k)) - \sum_{j\in\varrho}(f(x_j) - f(y_j)) \right| = m(T)$$

<u>Lemma 6</u> f is an isotone measure function for a set S on $[a,b]$ iff $\exists$ an antitone measure function g, for S, on $[a,b]$.

<u>Proof</u>: $\forall\, x \in [a,b]$ define $g(x) = f(b - x + a)$.
Now $x \in [a,b] \Rightarrow b - x + a \in [a,b] \Rightarrow f(b - x + a) \in \overline{E}_s \Rightarrow g(x) \in \overline{E}_s \Rightarrow g([a,b]) \subset \overline{E}_s$.
$y_0 \in \overline{E}_s$ and $f([a,b]) = \overline{E}_s \Rightarrow \exists\, x_0 \in [a,b] \ni f(x_0) = y_0$. Now $a \leq x_0 \leq b \Rightarrow \exists\, x_1 \in [a,b] \ni x_0 = b - x_1 + a \Rightarrow g(x_1) = y_0 \Rightarrow g([a,b]) \supset \overline{E}_s$
Therefore, $g([a,b]) = \overline{E}_s$.
$x_1 < x_2 \Rightarrow b - x_1 + a > b - x_2 + a \Rightarrow f(b - x_1 + a) \geq f(b - x_2 + a)$ (since f is isotone) $\Rightarrow g(x_1) \geq g(x_2)$
Then, by definition 17, g is an antitone measure function for S on $[a,b]$. By reversing the proof if g(x) is antitone then $f(b - x + a)$ is isotone.

Lemma 7 If f is a measure function for S on $[a,b]$, and $[c,d]$ is any bounded interval $\subset R$, then $\exists$ a measure function for S, on $[c,d]$.

Proof: $\forall x \in [c,d]$ define $g(x) = (x - d)a/(c - d) + (x - c)b/(d - c)$ and $\forall x \in [c,d]$ define the composite function $f \circ g(x) = f(g(x))$. $x \in [c,d] \Rightarrow g(x) \in [a,b] \Rightarrow f(g(x)) \in \bar{E}_S \Rightarrow f \circ g([c,d]) \subset \bar{E}_S$. $y_0 \in \bar{E}_S$ and $f([a,b]) = \bar{E}_S \Rightarrow \exists z_0 \in [a,b] \ni f(z_0) = y_0$. Now $\forall z_0 \in [a,b] \; \exists x_0 \in [c,d] \ni z_0 = (x_0 - d)a/(c - d) + (x_0 - c)b/(d - c)^9 \Rightarrow f \circ g(x_0) = y_0 \Rightarrow f \circ g([c,d]) \supset \bar{E}_S \Rightarrow f \circ g([c,d]) = \bar{E}_S$. $x_1 < x_2 \Rightarrow g(x_1) < g(x_2)$, so if f is antitone $f \circ g$ is antitone and if f is isotone $f \circ g$ is isotone. Hence, $f \circ g$ is a measure function for S, on $[c,d]$.

The measure of case I sets is additive.

Theorem 6 If S_1 and S_2 are Ⓜ case I sets, $S_1 \cup S_2$ is of case I, and $S_1 \cap S_2 = \emptyset$ then $S_1 \cup S_2$ is Ⓜ and $m(S_1 \cup S_2) = m(S_1) + m(S_2)$.

Proof: By lemma 6 $\exists$ isotone functions f_1 and $f_2 \ni f_1$ is a measure function for S_1 on $[a,b]$, and f_2 is a measure function for S_2 on $[c,d]$. By lemma 7 we can choose $[c,d] \ni b = c$. Then $\exists$ an isotone measure function

$$f_3(x) = \begin{cases} f_1(x) & \text{if } a \leq x \leq b \\ f_1(b) - f_2(c) + f_2(x) & \text{if } c \leq x \leq d \end{cases}$$

for $S_1 \cup S_2$ on $[a,d] \Rightarrow S_1 \cup S_2$ is Ⓜ.

If f_2 is a measure function for S_2 then, by theorem 5, $f_1(b) - f_2(c) + f_2(x)$ is a measure function for S . Since $f_3(x) \in \overline{\Pi}_{S_1}$ iff $f_3(x) \leq f_1(b)$ and $f_3(x) \in \overline{E}_S$ iff $f_3(x) \geq f_1(b)$, it is convenient to partition

$$\sum_{i \in \alpha_3}(f_3(x_i) - f_3(yi)) \text{ into } \sum_{i \in \alpha_1}^{f_3 \leq f_1(b)}(f_3(x_i) - f_3(y_i)) + \sum_{i \in \alpha_2}^{f_3 \geq f_1(b)}(f_3(x_i) - f_3(y_i))$$

and to partition γ_3 and ϱ_3 in the same way. This justifies:

$$m(S_1 \cup S_2) = \left| \sum_{i \in \alpha_3} F_i + \sum_{k \in \gamma_3} F_k - \sum_{j \in \varrho_3} F_j \right| = \left| \sum_{i \in \alpha_1}^{f_3 \leq f_1(b)} F_i + \sum_{i \in \alpha_2}^{f_3 \geq f_1(b)} F_i + \sum_{k \in \gamma_1}^{f_3 \leq f_1(b)} F_k + \sum_{k \in \gamma_2}^{f_3 \geq f_1(b)} F_k - \sum_{j \in \varrho_1}^{f_3 \leq f_1(b)} F_j - \sum_{j \in \varrho_2}^{f_3 \geq f_1(b)} F_j \right| = \left| \sum_{i \in \alpha_1}^{f_3 \leq f_1(b)} F_i + \sum_{k \in \gamma_1}^{f_3 \leq f_1(b)} F_k - \sum_{j \in \varrho_1}^{f_3 \leq f_1(b)} F_j \right| + \left| \sum_{i \in \alpha_2}^{f_3 \geq f_1(b)} F_i + \sum_{k \in \gamma_2}^{f_3 \geq f_1(b)} F_k - \sum_{j \in \varrho_2}^{f_3 \geq f_1(b)} F_j \right| \text{(by lemma 5)}$$

$$= \left| \sum_{i \in \alpha_1}(f_1(x_i) - f_1(y_i)) + \sum_{k \in \gamma_1}(f_1(x_k) - f_1(y_k)) - \sum_{j \in \varrho_1}(f_1(x_j) - f_1(y_j)) \right| + \left| \sum_{i \in \alpha_2}(f_2(x_i) - f_2(y_i)) + \sum_{k \in \gamma_2}(f_2(x_k) - f_2(y_k)) - \sum_{j \in \varrho_2}(f_2(x_j) - f_2(y_j)) \right|$$

(Since $f_3(x) \leq f_1(b) \Rightarrow f_3(x) = f_1(x)$ and $f_3(x) \geq f_1(b)$, $f_3(y) \geq f_1(b) \Rightarrow f_3(x) - f_3(y) = f_1(b) - f_2(c) + f_2(x) - (f_1(b) - f_2(c) + f_2(y)) = f_2(x) - f_2(y)$) $= m(S_1) + m(S_2)$.

<u>The measure of case I sets is finitely additive.</u>

<u>Corollary</u> If each of a set $S = \{ S_1, S_2, \ldots, S_n \}$ of mutually disjoint case I sets is Ⓜ and ✱ subset

$\{S_a, S_b, \ldots, S_h\} \subset S$, $\cup\{S_a, S_b, \ldots, S_h\}$ is of case I, then $\bigcup_{r=1}^{n} S_r$ is Ⓜ and $m(\bigcup_{r=1}^{n} S_r) = m(S_1) + m(S_2) + \ldots + m(S_n)$

Proof: By lemma 7 we can choose bounded intervals $[a_1, b_1], [a_2, b_2], \ldots, [a_n, b_n] \ni b_1 = a_2$, $b_2 = a_3$, $\ldots$, $b_{n-1} = a_n$ and by lemma 6 $\forall r \leq n$ $\exists$ an isotone measure function f_r for S_r on $[a_r, b_r]$. Then $\exists$ an isotone measure function:

$$f(x) = \begin{cases} f_1(x)\ , \text{ if } a_1 \leq x \leq b_1 \\ f_1(b_1) - f_2(a_2) + f_2(x)\ , \text{ if } a_2 \leq x \leq b_2 \\ f_1(b_1) - f_2(a_2) + f_2(b_2) - f_3(a_3) + f_3(x), \text{ if } a_3 \leq x \leq b_3 \\ \cdot \\ \cdot \\ \cdot \\ f_1(b_1) - f_2(a_2) + \ldots + f_{n-1}(b_{n-1}) - f_n(a_n) + f_n(x)\ , \text{ if } a_n \leq x \leq b_n \end{cases}$$

By theorem 6 $m(S_1 \cup S_2) = m(S_1) + m(S_2)$. $\forall i < n$ if $m(\bigcup_{r=1}^{i} S_r) = m(S_1) + m(S_2) + \ldots + m(S_i)$ then, by theorem 6, $\bigcup_{r=1}^{i} S_r$ and S_{i+1} are each Ⓜ case I sets and $(\bigcup_{r=1}^{i} S_r) \cup S_{i+1}$ is a case I set $\Rightarrow$ $m((\bigcup_{r=1}^{i} S_r) \cup S_{i+1}) = m(\bigcup_{r=1}^{i} S_r) + m(S_{i+1}) = m(S_1) + m(S_2) + \ldots + m(S_i) + m(S_{i+1})$ $\Rightarrow$ $m(\bigcup_{r=1}^{n} S_r) = m(S_1) + m(S_2) + \ldots + m(S_n)$ by finite induction.

Lemma 8 If S_1 and S_2 are Ⓜ case I sets, then $S_1 \supset S_2$ $\Rightarrow$ $m(S_1) \geq m(S_2)$

Proof: Since $S_1 \supset S_2$ then $I \in \mathcal{A}_{S_2} \Rightarrow I \in \mathcal{A}_{S_1}$. Hence, $m(\cup \mathcal{A}_{S_2}) = \sum_{i \in \alpha_2} m(I_i) \leq \sum_{i \in \alpha_1} m(I_i) = m(\cup \mathcal{A}_{S_1})$ (since the measure of disjoint intervals is countably additive). Now $K \in \mathcal{K}_{S_2} \Rightarrow K \in \mathcal{K}_{S_1}$ or K can be partitioned into subintervals each of which is an element of $\mathcal{K}_{S_1}$ or $\mathcal{J}_{S_1}$. In either case $m(K \cap E_{S_1}) \geq m(K \cap E_{S_2})$. Then, by theorems 2 and 6, $m(S_1) \geq m(S_2)$.

The measure of case I sets is countably additive.

Theorem 7 If $S_1, S_2, \ldots$ are mutually disjoint Ⓜ case I sets, $S = \bigcup_{r=1}^{\infty} S_r$ is a bounded case I set, and $\forall$ finite subset $\{S_a, S_b, \ldots, S_h\} \subset S$, $\bigcup\{S_a, S_b, \ldots, S_h\}$ is of case I, then S is Ⓜ and $m(S) = \sum_{r=1}^{\infty} m(S_r)$.

Proof: Since S is bounded we can define $[\text{g.l.b.}(E_S), \text{l.u.b.}(E_S)] = [u, v]$. Clearly $E_S \supset \bigcup_{i=1}^{\infty} E_S$

Now $\forall i \in \omega$ $S_i - E_{S_i}$ is at most enumerable[2] $\Rightarrow$

$\bigcup_{i=1}^{\infty}(S_i - E_{S_i}) = S - \bigcup_{i=1}^{\infty} E_{S_i}$ is at most enumerable[7] $\Rightarrow$

$S - \bigcup_{i=1}^{\infty} E_{S_i}$ has no points of condensation $\Rightarrow$ $E_S = \bigcup_{i=1}^{\infty} E_{S_i}$

$\Rightarrow$ each $E_{S_i} \subset [u, v]$.

$\forall x \in [u, v]$ define the sequence of functions:

$$f_1(x) = \begin{cases} u, & \text{if } [u,x] \cap E_{S_1} = \emptyset \\ \text{l.u.b.}([u,x] \cap E_{S_1}), & \text{if } [u,x] \cap E_{S_1} \neq \emptyset \end{cases}$$

$$f_2(x) = \begin{cases} u, & \text{if } [u,x] \cap (\bigcup_{i=1}^{2} E_{S_i}) = \emptyset \\ \text{l.u.b.}([u,x] \cap (\bigcup_{i=1}^{2} E_{S_i})), & \text{if } [u,x] \cap (\bigcup_{i=1}^{2} E_{S_i}) \neq \emptyset \end{cases}$$

.
.
.

$$f_n(x) = \begin{cases} u, & \text{if } [u,x] \cap (\bigcup_{i=1}^{n} E_{S_i}) = \emptyset \\ \text{l.u.b.}([u,x] \cap (\bigcup_{i=1}^{n} E_{S_i})), & \text{if } [u,x] \cap (\bigcup_{i=1}^{n} E_{S_i}) \neq \emptyset \end{cases}$$

.
.
.
.

If $r > s$ then $\forall x \in [u,v]$, $f_r(x) \geq f_s(x) \Rightarrow$ this sequence of isotone functions is isotone. This isotone sequence is bounded from above (by $g(x) = v$, $\forall x \in [u,v]$) and hence converges. Then the limit function $f(x)$ exists. Since $\bigcup_{i=1}^{\infty} E_{s_i} = E_s \Rightarrow [u,x] \cap (\bigcup_{i=1}^{\infty} E_{s_i}) = [u,x] \cap E_s$, then:

$$\lim_{n\to\infty} f_n(x) = f(x) = \begin{cases} u & \text{, if } [u,x] \cap E_s = \emptyset \\ \text{l.u.b.}([u,x] \cap E_s), & \text{if } [u,x] \cap E_s \neq \emptyset \end{cases}$$

By construction every function in this sequence of functions is isotone. If $f(x)$ is not isotone $\exists\, x_1, x_2 \in [u,v] \ni x_1 < x_2$ and $f(x_1) > f(x_2)$. Let $\epsilon_0 = (f(x_1) - f(x_2))/2$. Then $\lim_{n\to\infty} f_n(x_1) = f(x_1) \Rightarrow \exists\, N \in \omega \ni f(x_1) - f_N(x_1) < \epsilon_0$. Now, since $f(x_1) - f(x_2) > 0$, $f(x_1) - f(x_2) > (f(x_1) - f(x_2))/2 = \epsilon_0 > f(x_1) - f_N(x_1)$ $\Rightarrow f(x_1) - f(x_2) > f(x_1) - f_N(x_1) \Rightarrow f_N(x_1) > f(x_2)$. Now the sequence of functions is isotone $\Rightarrow f(x_2) \geq f_N(x_2)$ and $f_N(x_1) > f(x_2) \geq f_N(x_2) \Rightarrow f_N(x_1) > f_N(x_2)$ but $x_1 < x_2 \Rightarrow f_N$ is not isotone which is impossible. Therefore, f is isotone and $f([u,v]) = \overline{E}_s \Rightarrow f$ is a measure function for $S \Rightarrow S$ is ⓜ.

By the corollary to theorem 6, $m(\bigcup_{r=1}^{n} S_r) = m(S_1) + m(S_2) + \ldots + m(S_n)$. If $\lim_{n\to\infty} m(\bigcup_{r=1}^{n} S_r) > \lim_{n\to\infty}(\sum_{r=1}^{n} m(S_r))$ then $\exists\, N \in \omega \ni m(\bigcup_{r=1}^{N} S_r) > \lim_{n\to\infty}(\sum_{r=1}^{n} m(S_r))$. By lemma 8, $\lim_{n\to\infty}(\sum_{r=1}^{n} m(S_r)) \geq \sum_{r=1}^{N+1} m(S_r)$ and hence, $m(\bigcup_{r=1}^{N} S_r) > \sum_{r=1}^{N+1} m(S_r)$ which is impossible. Now if $\lim_{n\to\infty} m(\bigcup_{r=1}^{n} S_r) < \sum_{r=1}^{\infty} m(S_r)$, then $\exists\, N \in \omega \ni \lim_{n\to\infty} m(\bigcup_{r=1}^{n} S_r) < \sum_{r=1}^{N} m(S_r)$. But lemma 8 $\Rightarrow$ $m(\bigcup_{r=1}^{N+1} S_r) \leq \lim_{n\to\infty} m(\bigcup_{r=1}^{n} S_r)$, which $\Rightarrow \sum_{r=1}^{N+1} m(S_r) < \sum_{r=1}^{N} m(S_r)$ a contradiction. $m(S) = \lim_{n\to\infty} m(\bigcup_{r=1}^{n} S_r) = \sum_{r=1}^{\infty} m(S_r)$.

<u>Theorem 8</u> Every bounded set, S, of case I is $\textcircled{m}$.

<u>Proof:</u> $\exists$ a closed interval M, as defined in definition 6, and a monotone function, $\forall x \in [u,v] = M$

$$f(x) = \begin{cases} u \quad , \text{ if } [u,x] \cap E_s = \emptyset \\ \text{l.u.b.}([u,x] \cap E_s) \text{ , if } [u,x] \cap E_s \neq \emptyset \end{cases}$$

$f(M) = E_s$. Hence, S is $\textcircled{m}$ by definition 15 .

<u>Corollary 1</u> Any bounded open set, S, is $\textcircled{m}$.

<u>Proof:</u> S is open $\Rightarrow$ $S = S^{\circ} \Rightarrow \forall x \in S$, $\exists$ an open interval $U_x \ni x \in U_x \subset S$. Then $\bigcup_{x \in S}\{U_x\} = S$ and by Lindelöfs' theorem[10] $\exists$ an at most enumerable subset $\{U_i\} \subset \{U_x : x \in S\} \ni \bigcup_i \{U_i\} = S \Rightarrow$ the set $\{U_j : U_j \text{ is a maximal interval} \subset S\}$ is at most enumerable. Now $\bigcup_j \{U_j\} = S \Rightarrow S$ is of case I $\Rightarrow$ S is $\textcircled{m}$ by theorem 8 .

<u>Corollary 2</u> Any bounded closed set, S, is $\textcircled{m}$.

<u>Proof:</u> If S is not a case I set, then $\exists$ an interval, I, $\ni$ every subinterval of I contains a non-enumerable number of components of both E_s and $C(E_s)$. Hence, $x \in E_s \cap I \Rightarrow \{x\}$ is a component of E_s and $y \in C(E_s) \cap I \Rightarrow \{y\}$ is a component of $C(E_s)$[5]. Since $S - E_s$ is at most enumerable, then each subinterval of I contains a non-enumerable number of $C(S)$ each of which is a limit point of E_s (since E_s is everywhere dense) $\Rightarrow$ $\exists\, y \in C(S) \ni y \in \{\text{the closure of } S\} \Rightarrow S$ is not closed. Hence S is of case I and $\therefore$ $\textcircled{m}$ by theorem 8 .

<u>Theorem 9</u> If S and T are case I sets and $E_{S\cap T} \neq \emptyset$, then $S\cap T$ is of case I .

<u>Proof</u>: If $S\cap T$ is not a case I set, then $\exists$ an interval, I, $\ni$ every subinterval of I contains a non-enumerable number of components of both $E_{S\cap T}$ and $C(E_{S\cap T})$. The components of $E_{S\cap T}$ are everywhere non-enumerably dense in I $\Rightarrow$ the components of both E_s and E_t are everywhere dense in I and either E_s or E_t, say E_s, has a non-enumerable number of components in some subinterval $I_0 \subset I \Rightarrow$ every component of $C(E_s)\cap I_0$ is a single point. Since S is of caseI the number of components of $C(E_s)\cap I_0$ is at most enumerable. Then the number of components of $C(E_{S\cap T})\cap I_0$ is non-enumerable $\Rightarrow$ the number of components of $C(E_t)\cap I_0$ is non-enumerable in every subinterval of $I_0 \Rightarrow$ the number of components of $E_t\cap I_0$ is at most enumerable in every subinterval of I_0 . Now if the number of components of $E_{S\cap T}$ is non-enumerable then at least one component of $E_t \cap I_0$ is an interval $I_1 \subset I_0$. But $I_1 \cap C(E_t) = \emptyset$ and $I_1 \cap C(E_s)$ is at most enumerable $\Rightarrow \exists$ a subinterval $I_1 \subset I \ni$ the number of components of $C(E_{S\cap T})$ is at most enumerable. Therefore, $S\cap T$ is a case I set.

<u>Corollary</u> If $S_1, S_2, \ldots, S_n$ are caseI sets and $E_{\bigcap_{i=1}^{n} S_i} \neq \emptyset$, then $\bigcap_{i=1}^{n} S_i$ is of case I .

<u>Proof</u>: By theorem 9 and finite induction.

CHAPTER III

Case II Sets

Definition 19 A bounded set is of case II iff it is not of case I.

By theorem 8 all case I sets are Ⓜ . With this in mind we extend the definition of measurability to include case II sets.

Definition 20 A bounded set S (of case I or case II) is Ⓜ iff $\exists$ an at most enumerable set, $\{S_i\}$, of mutually disjoint case I sets $\ni \bigcup_i S_i = S$

By theorem 7 the measure of case I sets is countably additive. Since we desire the extended definition of measure to be countably additive we define:

Definition 21 Let S be a Ⓜ set with the set, $\{S_i\}$, of mutually disjoint case I sets $\ni \bigcup_i S_i = S$.
Then $m(S) = \sum_i m(S_i)$.

The measure of bounded sets is unique.

Theorem 10 If $\{S_i\}$ and $\{T_j\}$ are each a set of mutually disjoint case I sets $\ni \bigcup_i S_i = S$ and $\bigcup_j T_j = S$ then $\sum_i m(S_i) = \sum_j m(T_j)$.

Proof: Since each S_i and each T_j is a case I set, then, by theorem 9, $S_i \cap T_j$ is either at most enumerable or a case I set. Either condition $\Rightarrow$ $S_i \cap T_j$ is Ⓜ .

Now since $\bigcup_i S_i = \bigcup_j T_j$, then $\forall i \ S_i = S_i \cap (\bigcup_j T_j) = \bigcup_j (S_i \cap T_j)$. $\{S_i\}$ and $\{T_j\}$ are each at most enumerable $\Rightarrow$ the number of addends in $\sum_i \sum_j m(S_i \cap T_j)$ is at most enumerable[7]. $\forall i,j \ m(S_i \cap T_j) \geq 0 \Rightarrow \sum_i \sum_j m(S_i \cap T_j)$ is absolutely convergent $\Rightarrow \sum_i \sum_j m(S_i \cap T_j) = \sum_j \sum_i m(S_i \cap T_j)$[11]. Therefore $\sum_i m(S_i) = \sum_i \sum_j m(S_i \cap T_j) = \sum_j \sum_i m(S_i \cap T_j) = \sum_j m(T_j)$.

<u>The measure of bounded sets is countably additive</u>.

<u>Theorem 11</u> If $S_1, S_2, \ldots$ are mutually disjoint sets, then $\bigcup_{i=1}^{\infty} S_i = S$ is Ⓜ and $m(S) = \sum_{i=1}^{\infty} m(S_i)$

<u>Proof</u>: $\forall i \in \omega \ S_i$ is a Ⓜ bounded set $\Rightarrow \exists$ a set, $\{S_{r,i}\}$ of mutually disjoint case I sets $\ni \bigcup_r \{S_{r,i}\} = S_i \Rightarrow \exists$ a set $\{S_{r,i} : r,i \in \omega\}$ of mutually disjoint case I sets, which is enumerable[7], and hence $\bigcup_{i=1}^{\infty} (\bigcup_r \{S_{r,i} : r,i \in \omega\}) = S \Rightarrow S$ is Ⓜ, and $m(S) = \sum_{i=1}^{\infty} \sum_r m(S_{r,i}) = \sum_{i=1}^{\infty} m(S_i)$.

<u>The measure of bounded sets is translation invariant</u>.

<u>Theorem 12</u> If S is a Ⓜ bounded set and $\exists c \in R \ni T = \{t : t = y + c, y \in S\}$, then T is Ⓜ and $m(S) = m(T)$.

<u>Proof</u>: If S is of case I then the theorem is true by theorem 5 . If S is of caseII then $\exists$ a set $\{S_i\}$ of mutually disjoint case I sets $\ni \bigcup_i \{S_i\} = S$.

Each S_i is of case I $\Rightarrow$ $\forall i$ $S_i(c) = \{x : x = y + c,\ y \in S_i\}$ is of case I $\Rightarrow$ $\exists$ a set $\{S_i(c)\}$ of mutually disjoint case I sets $\ni \bigcup_i S_i(c) = T$ $\Rightarrow$ T is (m). By theorem 5, $\forall i$ $m(S_i) = m(S_i(c))$. Hence, $m(S) = \sum_i m(S_i) = \sum_i m(S_i(c)) = m(T)$.

<u>Theorem 13</u> $\exists$ a (m) case II set.

<u>Proof</u>: The construction of this set is an elaboration of a set given by Hobson[12].

In the following discussion $\forall i \in \omega$, $c_i \in \{0,2\}$, $a_i \in \{0,1,2\}$ and let $(b_1, b_2, \ldots, b_n, \ldots) = b_1/3 + b_2/3^2 + \ldots + b_n/3^n + \ldots$

$C = \{x : x = (c_1, c_2, \ldots, c_n, \ldots)\}$ which is the Cantor discontinuum. Let $A_1 = \{x : x = (1, c_2, c_3, \ldots, c_n, \ldots)$, where not all $c_i = 0$ nor all $c_i = 2\}$ which is the Cantor discontinuum in the interval $(1/3, 2/3)$.

We now define the sequence $\{S_i\}$ of case I sets $\ni$ $\bigcup_{i=1}^{\infty} S_i = S$.

Let $S_1 = \{x : x = (a_1, c_2, c_3, \ldots, c_n, \ldots)\} = C \cup \{x : x = (1, c_2, c_3, \ldots, c_n, \ldots)$, where not all $c_i = 0$ nor all $c_i = 2\}$ $= C \cup A_1$. Clearly $m(S_1) = m(C) + m(A_1) = 0 + 0 = 0$, since A_1 is the Cantor discontinuum in the interval $(1/3, 2/3) \Rightarrow$ $m(A_1) = 0$ and $A_1 \cap C = \emptyset$.

Let $S_2 = \{x : x = (a_1, a_2, c_3, c_4, \ldots, c_n, \ldots)\} = C \cup A_1 \cup \{x : x = (0, 1, c_3, c_4, \ldots, c_n, \ldots)\} \cup \{x : x = (2, 1, c_3, c_4, \ldots, c_n, \ldots)\} \cup \{x : x = (1, 1, c_3, c_4, \ldots, c_n, \ldots)\} =$ $C \cup A_1 \cup A_2 \cup A_3 \cup A_4$ where not all $c_i = 0$, nor all $c_i = 2$

in any A_i, and A is the Cantor discontinuum in the interval (1/9, 2/9) , A_3 is the Cantor discontinuum in the interval (7/9, 8/9) , A_4 is the Cantor discontinuum in the interval (4/9, 5/9) . Clearly $m(S_2) = m(C) + m(A_1) + m(A_2) + m(A_3) + m(A_4) = 0$.

Let $S_3 = \{ x : x = (a_1, a_2, a_3, c_4, c_5, \ldots, c_n, \ldots) \} = C \cup A_1 \cup A_2 \cup A_3 \cup A_4 \cup \{ x : x = (0,0,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (0,1,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (0,2,1,c_4,c_5,\ldots c_n,\ldots) \} \cup \{ x : x = (1,0,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (1,1,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (1,2,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (2,0,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (2,1,1,c_4,c_5,\ldots,c_n,\ldots) \} \cup \{ x : x = (2,2,1,c_4,c_5,\ldots) \}$

In the same way $m(S\ ) = 0$.

.
.
.

Let $S_n = \{ x : x = (a_1, a_2, \ldots, a_n, c_{n+1}, c_{n+2}, \ldots, c_{n+m}, \ldots) \} = C \cup A_1 \cup A_2 \cup \ldots \cup A_{\frac{3^n-1}{2}}$ and since each A_i is a Cantor discontinuum, $m(S_n) = 0$.

.
.
.

Now $m > n \Rightarrow S_m \supset S_n$. Hence $x \in$ upper limit$\{S_n\} \Rightarrow x \in$ lower limit$\{S_n\} \Rightarrow$ upper lim$\{S_n\} \subset$ lower lim$\{S_n\} \Rightarrow$ upper lim$\{S_n\}$ = lower lim$\{S_n\} \Rightarrow \exists \lim_{n\to\infty} \{S_n\}$[13].

Therefore, let $S = \lim_{n\to\infty}\{S_n\}$. It is clear that:

$S = \{ s : s = (a_1, a_2, \ldots, a_n, \ldots)$, where $\exists N \in \omega \ni n > N \Rightarrow a_n \neq 1 \}$. In other words, in the triadic expansion of s, there are a finite number of $a_i = 1$.

By the fact that S = a countable union of mutually disjoint case I sets, $C \cup (\bigcup_{i=1}^{\infty} A_i)$, each of which is of measure zero, it is true that $m(S) = 0$.

Every point of S is a point of condensation of S (since every point of S is an element of some perfect set). By construction $S = E_s$ is everywhere dense. S is the countable union of nowhere dense perfect sets $\Rightarrow$ C(S) is the countable intersection of everywhere dense open sets $\Rightarrow$ C(S) is everywhere dense in $[0,1]$ [14]. Since C(S) has no isolated points , $\forall$ subinterval $I \subset [0,1]$, $I \cap C(S)$ is non-enumerable[15]. Clearly $x \in S \Rightarrow \{x\}$ is a component of E_s and $y \in C(S) \Rightarrow \{y\}$ is a component of $C(E_s)$, Therefore, each subinterval $I \subset [0,1]$ contains a non-enumerable number of components of $E_s = S$ and $C(E_s) = C(S)$. Hence, S is a case II set.

CHAPTER IV

Comparison with Lebesgue Measure

Measure as defined in this paper will be referred to as $\textcircled{m}$-measure and Lebesgue measure will be referred to as μ-measure.

<u>Theorem 14</u> If a bounded set S is $\textcircled{m}$-measurable, then it is μ-measurable and $m(S) = \mu(S)$.

<u>Proof:</u> $\forall$ interval I, $m(I) = \text{l.u.b.}(I) - \text{g.l.b.}(I)$ (by lemma 3) $= \mu(I)$. <u>If</u> S <u>is</u> <u>of</u> <u>case</u> I then: $m(S) = \sum_{i\in\alpha}|F_i| + \sum_{k\in\nu}|F_k| - \sum_{j\in\varrho}|F_j|$ (by definition 16 and lemma 5 $= \sum_{i\in\alpha} m(I_i) + \sum_{k\in\nu} m(K_k) - \sum_{j\in\varrho} m(J_j) = \sum_{i\in\alpha}\mu(I_i) + \sum_{k\in\nu}\mu(K_k) - \sum_{j\in\varrho}\mu(J_j) = \mu(\{\cup\mathcal{A}_s\}) + \mu(\{\cup\mathcal{K}_s\}) - \mu(\{\cup\mathcal{J}_s\})$ (since μ-measure is countably additive[16]) $= \mu(\{\cup\mathcal{A}_s\}\cup\{\cup\mathcal{K}_s\}) - \mu(\{\cup\mathcal{J}_s\}) = \mu(\{\cup\mathcal{A}_s\}\cup\{\cup\mathcal{K}_s\} - \{\cup\mathcal{J}_s\})$ [17] $= \mu(S)$ (since, by theorem 3, $E_s - [\{\cup\mathcal{A}_s\}\cup\{\cup\mathcal{K}_s\} - \{\cup\mathcal{J}_s\}]$ is at most enumerable and $S - E_s$ is at most enumerable[2], each is of μ-measure zero[18]) .

If S is of case II then $\exists$ an at most enumerable set of mutually disjoint case I sets $\{S_i\} \ni \bigcup_i S_i = S$. Now $m(S) = \sum_i m(S_i) = \sum_i \mu(S_i)$ (since each S_i is of case I) $= \mu(S)$ (since μ-measure is countably additive[16]) .

This also proves that if a bounded set ,S, is μ-measurable and S is $\textcircled{m}$-measurable then $\mu(S) = m(S)$.

<u>Theorem 15</u> If a bounded set S is μ-measurable, then $\exists$ a set $T \subset S \ni m(T) = \mu(S)$.

<u>Proof</u>: S is μ-measurable $\Rightarrow$ $\exists$ a sequence $\{S_i\}$ of closed sets $\ni \mu(S - \bigcup_{i=1}^{\infty} S_i) = 0$[19] $\Rightarrow$ $\mu(S) - \mu(\bigcup_{i=1}^{\infty} S_i) = 0$[17] $\Rightarrow$ $\mu(S) = \mu(\bigcup_{i=1}^{\infty} S_i)$. $\forall\, i \in \omega$ S_i is a closed set $\Rightarrow$ S_i is $\textcircled{m}$-measurable (by the second corollary to theorem 8) $\Rightarrow$ $\bigcup_{i=1}^{\infty} S_i$ is $\textcircled{m}$-measurable (by theorem 11) $\Rightarrow$ $\mu(\bigcup_{i=1}^{\infty} S_i) = m(\bigcup_{i=1}^{\infty} S_i)$ (by theorem 14) . Therefore, $\mu(S) = m(T)$, where $T = \bigcup_{i=1}^{\infty} S_i \subset S$.

CHAPTER V

Non-measurable Sets

Definition 22 A set is of the first category iff it is the union of a countable number of nowhere dense sets, and a set is of the second category iff it is not of the first category.

Definition 23 A set, S, is of the second category at a point x iff $\forall$ neighborhood U of x $U \cap S$ is of the second category[20].

Theorem 16 If $\exists$ an interval I $\ni$ a set S is of the second category at every point of I and C(S) is of the second category at every point of I, then S is non-Ⓜ-measurable.

Proof: $\forall\, x \in S \cap I$ and $\forall$ neighborhood U of x, $U \cap S$ is non-enumerable (if $U \cap S$ is enumerable then it is the union of a countable family, $\{x : x \in S \cap I\}$, of nowhere dense sets). Hence $x \in S \cap I \Rightarrow x \in E_s \cap I$. Similarly, $y \in C(S) \cap I \Rightarrow y \in C(E_s) \cap I$.

1) $x \in E_s \cap I \Rightarrow \{x\}$ is a component of $E_s \cap I$ (else an interval $I_1 \ni x \in I_1 \subset I \Rightarrow$ C(S) is nowhere dense in $I_1 \subset I$ which is impossible).

   Dually, $y \in C(E_s) \cap I \Rightarrow \{y\}$ is a component of $C(E_s) \cap I$.

Therefore, each subinterval of I contains a non-enumerable number of components of each E_s and $C(E_s)$ $\Rightarrow$ S is a case II set.

If S is ⓜ then $\exists$ an at most enumerable set, $\{ S_i \}$, of mutually disjoint case I sets $\ni \bigcup_i S_i = S$. $\forall S_n \in \{S_i\}$ S_n is of case I $\Rightarrow$ M_n (as defined in definition 6) can be partitioned into a set of mutually disjoint intervals $\{A_n\} \ni$ each $A_m \in \{A_n\}$ contains an at most enumerable number of components of either E_{s_n} or $C(E_{s_n})$. By 1) $\forall S_n \in \{ S_i \}$ each component of E_{s_n} is a single point (since no component of S can be an interval[5]) $\Rightarrow$ each A_m contains either no components of E_{s_n} or a non-enumerable number of components of E_{s_n} (since no interval can contain only an enumerable number of points of condensation of a set). If A_m contains a non-enumerable number of components of E_{s_n} then A_m contains an enumerable number of components of $C(E_{s_n})$ which includes an everywhere dense set of intervals (else $\exists$ a subinterval $I_1 \subset A_m \ni$ every component of $C(E_{s_n})$ contained in I_1 is a point and the number of components of $C(E_{s_n})$ contained in I_1 is at most enumerable $\Rightarrow$ $C(E_{s_n}) \cap I_1$ is an at most enumerable set of points $\Rightarrow$ $C(S_n) \cap I_1$ is an at most enumerable set of points $\Rightarrow$ $C(S) \cap I_1$ is an at most enumerable set of points $\Rightarrow$ C(S) is not of the second category at any interior point of $I_1 \subset I$ which is impossible) $\Rightarrow$ E_{s_n} is nowhere dense in A_m. Therefore, each E_{s_n} is nowhere dense in M. Now $S - \bigcup_n E_{s_n}$ is at most enumerable and hence $\exists$ a countable number of nowhere dense sets, whose union is S $\Rightarrow$ S is a first category set which is false. Therefore, S is non-$\mathfrak{m}$-measurable.

We now show, with the following three lemmas and a theorem, that non-$\mathfrak{m}$-measurable sets do exist.

Lemma 9 Density is preserved under a translation.

Proof: If a set U is dense in a set V then $\overline{U} \supset V$.
Let $r_0 \in R$, $U_1 = \{x + r_0 : x \in U\}$, $V_1 = \{x + r_0 : x \in V\}$
Clearly $x \in U \cap V \Rightarrow x + r_0 \in U_1 \cap V_1$. Since $\overline{U} \supset V$
$\forall x \in V \quad \exists$ a sequence $\{x_n\} \subset U \ni \{x_n\} \longrightarrow x \Rightarrow$
$\forall x + r_0 \in V_1 \quad \exists \{x_n + r_0\} \subset U_1 \ni \{x_n + r_0\} \longrightarrow x + r_0$
$\Rightarrow \overline{U}_1 \supset V_1 \Rightarrow U_1$ is dense in V_1.

Lemma 10 If a set V is of the second category at some point, then $\exists$ an interval I $\ni$ V is everywhere dense in I and V is of the second category at every point of I.

Proof: If $\nexists$ an interval I $\ni$ V is everywhere dense in I, then V is nowhere dense in [g.l.b.(V), l.u.b.(V)] $\Rightarrow$ V is of the first category at every point which is false.
If $\forall$ interval I where V is everywhere dense $\exists\, a \in I$ V is of the first category at a, then $U = \{a : a \in I,$ V is of the first category at a$\}$ is everywhere dense in I and $\forall a \in U \quad \exists$ a neighborhood O_a of a $\ni O_a \cap V$ is of the first category. Then $\bigcup_{a \in U} O_a \supset I$ and $V \cap (\bigcup_{a \in U} O_a)$ is of the first category $\Rightarrow$ $V \cap I$ is of the first category.
Since V is non-dense elsewhere then V is of the first category at all points of [g.l.b.(V), l.u.b.(V)].

Hence, $\exists$ an interval I $\ni$ V is not of the first category at any point of I $\Rightarrow$ V is of the second category at every point of I .

Lemma 11 The union of a finite number of sets $S_1, S_2, \ldots, S_n$ each of which is non-dense in the same interval I is non-dense in I .

Proof: Let I_0 be any interval contained in I . S_1 is non-dense in I $\Rightarrow$ $\exists$ a subinterval $I_1 \subset I_0 \ni S_1 \cap I_1 = \emptyset$ S_2 is non-dense in I $\Rightarrow$ S_2 is non-dense in I_1 $\Rightarrow$ $\exists$ a subinterval $I_2 \subset I_1 \ni S_2 \cap I_2 = \emptyset$. Proceeding in the same way $\exists$ a subinterval $I_n \subset I_{n-1} \subset \cdots \subset I_0 \ni$ $S_n \cap I_n = \emptyset$. By construction $\forall$ i, $1 \leq i \leq n$ $S_i \cap I_i = \emptyset$ and , therefore, $S_i \cap I_n = \emptyset$. Then $(\bigcup_{i=1}^{n} S_i) \cap I_n = \emptyset$ and hence, $\bigcup_{i=1}^{n} S_i$ is non-dense in I .

Theorem 17 $\exists$ a non-$\mathfrak{m}$-measurable set.

Proof: The following non-μ-measurable set is given by Kestelman[21]. Let $I_0 = (0,1)$ $\forall$ # $\in I_0$ let R(#) = $\{x : 0 < x < 1,\ x - \# \in Q$ = the set of rational numbers$\}$ Then $* \in R(\#) \Rightarrow R(\#) = R(*)$. Thus each two different such sets are disjoint. By Zermelo's Axiom $\exists$ a set S $\ni$ $S \subset I$ and $\forall x \in I_0$, $S \cap R(x)$ consists of a single point.

Since S is non-μ-measurable then, by theorem 14, S is non-$\mathfrak{m}$-measurable. Now let us prove this set,S, is non-$\mathfrak{m}$-measurable by considering its topological properties.

To Prove: S is of the second category at every point of some interval I .

Proof: If S is of the first category at all points then, $\exists$ a sequence $\{S_i\}$ of nowhere dense sets $\ni \bigcup_{i=1}^{\infty} S_i = S$ Let $r_1, r_2, \ldots, r_j, \ldots$ be an enumeration of the set of rational numbers between -1 and 1 . Let $S_i(r_j) = \{x + r_j : x \in S_i\}$. By lemma 9, S_i is non-dense $\Rightarrow$ $S_i(r_j)$ is non-dense. By lemma 11, $\forall\, n \in \omega$ $\bigcup_{j=1}^{n} S_n(r_j)$ is non-dense and so $\bigcup_{n=1}^{\infty}\left[\bigcup_{j=1}^{n} S_n(r_j)\right]$ is a countable union of nowhere dense sets and is, therefore, of the first category. But $\bigcup_{n=1}^{\infty}\left[\bigcup_{j=1}^{n} S_n(r_j)\right] \supset (0,1)$ which is a second category set, which contradicts the previous statement. Therefore, S is of the second category at at least one point. Now, by lemma 10, $\exists$ an interval I $\ni$ S is of the second category at every point of I .

To Prove: $\exists$ a subinterval $I_1 \subset I \ni C(S)$ is of the second category at every point of I_1 .

Proof: Let $V = \{x + r_x : x \in S,\ 0 \neq r_x \in Q,\ x + r_x \in I\}$ ($\exists\, V \neq \emptyset$, since Q is everywhere dense in R)

Let $r_1, r_2, \ldots, r_j, \ldots$ be an enumeration of the set of rational numbers between -1 and 1 . If V is of the first category at all points, then $\exists$ a sequence $\{V_i\}$ of non-dense sets $\ni \bigcup_{i=1}^{\infty} V_i = V$. Let $V_i(r_j) = \{x + r_j : x \in V_i\}$. Each $V_i(r_j)$ is non-dense by lemma 9 and each $\bigcup_{j=1}^{n} V_n(r_j)$ is non-dense by lemma 11 $\Rightarrow$

$\bigcup_{n=1}^{\infty}[\bigcup_{j=1}^{n} V_n(r_j)]$ is a first category set, but

$\bigcup_{n=1}^{\infty}[\bigcup_{j=1}^{n} V_n(r_j)] \supset (0,1)$ which is a second category set. Therefore, V is a second category set at some point and, by lemma 10, $\exists$ an interval $I_1 \ni$ V is of the second category at every point of I_1 .

By definition of S, $x \in S$ and $0 \neq r_x \in Q \Rightarrow x + r_x \notin S$

Hence, $V \subset C(S)$ and, therefore, C(S) is of the second category at every point of I_1 . Now $I_1 \subset I$ and S is of the category at every point of I $\Rightarrow$ S is of the second category at every point of I_1 .

Hence, $\exists$ an interval $I_1 \ni$ both S and C(S) are of the second category at every point of $I_1 \Rightarrow$ S is non-$\textcircled{m}$-measurable.

CHAPTER VI

Extensions

<u>Definition 24</u> An unbounded set, S, is Ⓜ iff $\exists$ an at most enumerable set $\{S_i\}$ of bounded mutually disjoint case I sets $\ni \bigcup_i S_i = S$.

<u>Definition 25</u> If an unbounded set, S, is Ⓜ , then an at most enumerable set $\{S_i\}$ of bounded mutually disjoint case I sets $\ni \bigcup_i S_i = S$, and $m(S) = \sum_i m(S_i)$

It is clear from the methods of proof in theorems 10, 11, and 12 that measure of unbounded sets is unique, translation invariant, and countably additive.

Definition 26 A set $S = S_1 X S_2 X \ldots X S_n \subset R^n$ is Ⓜ iff $\forall i \ni 1 \leq i \leq n$ S_i is Ⓜ in R_i

Definition 27 If a set $S \subset R^n$ is Ⓜ, then $m(S) = m(S_1) \cdot m(S_2) \cdot \ldots \cdot m(S_n)$

These definitions are presented to show that measure, as defined in this paper, is in no way restricted to bounded sets or one dimensional spaces.

REFERENCES

1. Kestelman, H. , Modern Theories of Integration, Dover New York, 1960, p. 27

Theorem 40.1 If S is a non-enumerable set, then S includes at least one point of condensation of S.

2. Kestelman, H. , op. cit. , p. 28

Corollary to Theorem 40.1 If S is any set, then the set of the points of S which are not limiting points of S is finite or enumerable.

3. Kelley, J.L. , General Topology, van Nostrand, Princeton, 1955, p. 38

Definition

4. Kelley, J.L. , op. cit. , p. 51

Definition

5. Mendelson, B. , Introduction to Topology, Allyn and Bacon, Boston, 1962, p. 124

Theorem 3.4 Apart from the empty set and single points the only connected subsets of the real line are intervals

6. Kelley, J.L. , op. cit. , p. 55

Theorem 22 Every connected subset of a topological space is contained in a maximal connected set called a component.

7. Kamke, E. , Theory of Sets, Dover, New York, 1950, p.8

Theorem 2 The union of enumerably many sets, each of which is at most enumerable, is at most enumerable.

8. McShane, E.J. and Botts, T.A. , Real Analysis, van Nostrand, Princeton, 1959, p. 58

Definition

9. McShane, E.J. and Botts, T.A. , op. cit. , p. 72

Method of the mapping given on this page

10. Kestelman, H. , op. cit. , p. 26

Theorem 39(Lindelof) $\forall$ non-empty open covering of a set S, $\exists$ a countable subcovering of S.

11. Knopp, K. , Infinite Sequences and Series, Dover, New York, 1956, p. 79

Theorem 4 For any absolutely convergent series, any rearrangement of it converges to the same value.

12. Hobson, E.W. , The Theory of Functions of a Real Variable and the Theory of Fourier's Series, Cambridge University Press, Cambridge, Vol. I, 1927, p. 135

13. Saks, S. , Theory of the Integral, Dover, New York, 1964, p. 5

Definition of the limit of a sequence of sets

14. Kelley, J.L. , op. cit. , p. 200

Theorem 34(Baire) If R is a complete pseudo-metric space, then the intersection of a countable family of open dense subsets of R is itself dense in R.

15. Saks, S. , op. cit. , p. 55

Theorem 9.4 In a complete metric space, every non-empty intersection of open sets, without isolated points, is non-enumerable.

16. Kestelman, H. , op. cit. , p. 73

Theorem 94 If A_r is a sequence of mutually disjoint measurable sets, then $m(\quad A_r) = \quad m(A_r)$.

17. Kestelman, H. , op. cit. , p. 74

Theorem 97 If A and B are measurable sets and A $\supset$ B, then A-B is measurable and $m(A-B) = m(A) - m(B)$

18. Kestelman, H., op. cit. , p. 72

Theorem 89 Every enumerable set is of measure zero.

19. Kestelman, H., op. cit. , p. 80

Theorem 111 If S is a measurable set then $\exists$ a sequence of closed sets $\{C_r\} \subset S \ni m(S - \bigcup_{r=1}^{\infty} C_r) = 0$

20. Kelley, J.L., op. cit. , p. 201

Definition

21. Kestelman, H., op. cit. , p. 90

Theorem 130 $\exists$ a non-measurable set.

22. Halmos, P.R., Measure Theory , van Nostrand, Princeton, 1950, p. 62

Discussion

23. Halmos, P.R., op. cit. , p. 70

Theorem E

24. Russell, B.A.W., An Essay on the Foundations of Geometry , Dover , New York , 1956 , p. 164

Discussion

25. Russell, B.A.W., op. cit. , p. 149

Discussion

BIBLIOGRAPHY

Halmos, P.R. , Measure Theory , van Nostrand , Princeton , 1950

Hobson, E.W. , The Theory of Functions of a Real Variable and The Theory of Fourier's Series , Cambridge University Press , Cambridge , Vol. I , 1927

Kamke, E. , Theory of Sets , Dover , New York , 1950

Kelley, J.L. , General Topology , van Nostrand , Princeton , 1955

Kestelman , H. , Modern Theories of Integration , Dover , New York , 1960

Knopp, K. , Infinite Sequences and Series , Dover , New York , 1956

McShane, E.J. , and Botts, T.A. , Real Analysis , van Nostrand , Princeton , 1959

Mendelson, B. , Introduction to Topology , Allyn and Bacon , Boston , 1962

Russell, B.A.W. , An Essay on the Foundations of Geometry , Dover , New York , 1956

Saks, S. , Theory of the Integral , Dover , New York , 1964