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# Delay-Independent Stability Analysis of Linear Time-Delay Systems Based on Frequency Discretization[☆]

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**Abstract**

This paper studies strong delay-independent stability of linear time-invariant systems. It is known that delay-independent stability of time-delay systems is equivalent to some *frequency-dependent* linear matrix inequalities. To reduce or eliminate conservatism of stability criteria, the frequency domain is discretized into several sub-intervals, and *piecewise constant* Lyapunov matrices are employed to analyze the frequency-dependent stability condition. Applying the generalized Kalman–Yakubovich–Popov lemma, new necessary and sufficient criteria are then obtained for strong delay-independent stability of systems with a single delay. The effectiveness of the proposed method is illustrated by a numerical example.

*Keywords:* Time-delay systems, frequency discretization, delay-independent stability, Lyapunov inequality.

## 1. Introduction

In many practical systems such as industrial processes and networked control systems, time-delay phenomena are inevitably encountered, and are often the key factor that affects the performance (Gu et al., 2003; Wang et al., 2015). Time-delay systems, although with a long history, are one of the most active topics in control and system theory in the past two decades, see Gu and Niculescu (2003); Sipahi et al. (2011) and the references therein. Even the most basic problem, stability analysis, of time-delay systems is still challenging due to its infinite-dimensional nature (Gu et al., 2003), and such study is still evolving (Sipahi et al., 2011). Sometimes, stability of systems can be maintained for all positive delays, thus giving the notion of *delay-independent* stability. This is in contrast to delay-dependent stability, in which case the system is stable for only certain range of delay values. In this paper, we focus on delay-independent stability.

The term "delay-independent stability" was introduced in Hale (1977), and many criteria have been developed for testing delay-independent stability of time-delay systems since then (see Delice and Sipahi (2012); Souza et al. (2009) for examples of more recent developments). Delay-independent stability itself includes two different notions, viz., *strong* delay-independent stability and *weak* delay-independent stability (see Definitions 1 and 2 in Section 2.1, respectively). The strong delay-independent stability, albeit being as a special case of the weak delay-independent one, is sufficiently general from a practical robustness point of view (Bliman, 2002). Necessary and sufficient criteria of delay-independent stability (both strong and weak) are often developed using a frequency domain method based on the characteristic equation. Some typical tools used include polynomial theory (Kamen, 1982), matrix pencil (Niculescu, 1998b), and robust control theory (Chen and Latchman, 1995). In addition to direct stability test, the necessary and sufficient conditions may also be useful in developing other simpler sufficient conditions that are easier to test, and uncovering their inherent conservatism.

A number of sufficient conditions for delay-independent stability can also be found in the literature (Boyd et al., 1994; Chen et al., 1995; Kolmanovskii et al., 1999). Although efforts in stability analysis are made mainly to derive necessary and sufficient conditions, the interest in some sufficient conditions are due to two factors. *First*, some sufficient conditions usually require much less computation than typical necessary and sufficient ones. *Second*, many sufficient conditions, especially those based on the Lyapunov stability theory (Boyd et al., 1994; Kolmanovskii et al., 1999), are easily adapted to other more complicated problems of time-delay systems. In fact, fruitful synthesis results on time-delay systems, whether delay-independent (Boyd et al., 1994; Shi et al., 1999; Wang et al., 1999; Wu and Grigoriadis, 2001) or delay-dependent (Du et al., 2010; Palhares et al., 2005; Fridman and Shaked, 2002; He et al., 2004; Lin et al., 2006; Li and Gao, 2011; Gao and Li, 2011), can be regarded as applications or extensions of simple linear matrix inequality (LMI) conditions (Boyd et al., 1994; Agathoklis and Foda, 1989).

In the paper, we will revisit the problem of strong delay-independent stability analysis of linear time-invariant systems with a state delay. Our attention will be focused on applying a *frequency-discretization* idea to develop new stability criteria in terms of linear matrix inequality (LMI). The advantage of the proposed stability criteria lies in the fact that they give a series of new sufficient conditions for systems with a single delay and become *nonconservative* as the frequency-discretization number goes to infinity, thus potentially less conservative than some typical sufficient LMI conditions in the literature. Numerical results will be provided to illustrate the improvement of the proposed method.

*Notation:* The superscripts "$-1$", "T", "$*$"and "$\perp$" stand for inverse, transpose, conjugate transpose and null space of a matrix, respectively. $\mathbb{R}^{m\times n}$ ($\mathbb{C}^{m\times n}$) is the set of $m\times n$ real (complex) matrices. $\mathbb{C}_+$ denotes the closed right half plane of the complex

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plane, and $\mathbb{D}$ and $\partial\mathbb{D}$ denote the closed unit disc and the unit circle on the complex plane, respectively. The notation $P > 0$ ($\geq 0$) means that matrix $P$ is Hermitian positive definite (semi-definite). $\mathbf{S}_n$ and $\mathbf{H}_n$ are the sets of $n \times n$ symmetric and Hermitian matrices, respectively. $\mathbf{I}$ denotes an identity matrix with appropriate dimension. For a square matrix $A$, $\text{sym}\{A\}$ represents $(A^* + A)/2$. For a square matrix $A$, $\alpha(A)$ and $\rho(A)$ are the spectral abscissa and spectral radius of $A$, respectively, $\lambda(A)$ and $\sigma(A)$ are the eigenvalues and singular values of $A$, respectively. Matrix dimensions are assumed to be compatible for algebraic operations.

## 2. Main Results

In this section, we present new stability conditions for systems with a single delay. Section 2.1 formulates the problem and provides some preliminaries. Section 2.2 comments some existing results for motivation. Technical details of the frequency-discretization idea and stability conditions are presented in Section 2.3, and numerical implementation of the stability conditions is discussed in Section 2.4.

### *2.1. Problem statement and preliminaries*

Consider a linear continuous time-invariant system with a single delay described by the following delay-differential equation:

$$\dot{x}(t) = A_0 x(t) + A_1 x(t-d), \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the state vector, $A_0$ and $A_1 \in \mathbb{R}^{n\times n}$ are known constant matrices, and $d \geq 0$ is the delay. Define a bivariate polynomial $c(s,z)$ as

$$c(s,z) \triangleq \det(s\mathbf{I} - A_0 - zA_1).$$

For a given delay $d$, it is known (Hale, 1977) that the asymptotic stability of system (1) is equivalent to

$$c(s,z) \neq 0,\ \forall s \in \mathbb{C}_+ \text{ and } z = \mathrm{e}^{-ds}. \tag{2}$$

In this paper, we are interested in system (1) whose stability is maintained for arbitrary delay $d \geq 0$. Two related notions of *delay-independent stability* for system (1) are defined as follows.

**Definition 1.** System (1) is said to be (weakly) delay-independently stable if the condition in (2) is satisfied for all $d \geq 0$.

**Definition 2.** System (1) is said to be strongly delay-independently stable if

$$c(s,z) \neq 0,\ \forall (s,z) \in \mathbb{C}_+ \times \mathbb{D}. \tag{3}$$

According to the definition, strong delay-independent stability is defined by regarding $s$ and $z$ as independent of each other. As emphasized in Chen and Latchman (1995), strong delay-independent stability is stricter than the weak version in terms of the requirement at $s = 0$, where $z = 1$ in $c(s,z)$ can no longer be regarded as a variable independent of $s$ any more. However, the property of weakly delay-independent stability is not robust against perturbations of parameters $A_0$ and $A_1$ (Bliman, 2002). In this paper, we mainly consider strong delay-independent stability (but see Remark 3).

It is difficult to test strong delay-independent stability of system (1) directly according to its definition, because $c(s,z)$ is a bivariate polynomial. Define

$$S(s) \triangleq (s\mathbf{I} - A_0)^{-1} A_1,\ Z(z) \triangleq A_0 + zA_1.$$

The condition in (3) can be simplified to overcome this difficulty.

**Lemma 1.** *System (1) is strongly delay-independently stable if and only if either one of the following two equivalent conditions holds.*

*(i)*

$$\rho(S(s)) < 1,\ \forall \Re(s) = 0 \tag{4}$$

*and*

$$\alpha(A_0) < 0. \tag{5}$$

*(ii)* $\alpha(Z(z)) < 0$ *for all* $z \in \partial\mathbb{D}$.

Condition (i) has been established in Chen and Latchman (1995), and Agathoklis and Foda (1989); and condition (ii) can be found in Kamen (1982), and Agathoklis and Foda (1989). In this paper, they will be used to develop novel and tractable stability criteria for system (1).

### *2.2. Observation and Motivation*

It has been well understood (Boyd et al., 1994; Agathoklis and Foda, 1989) that condition (i) of Lemma 1 holds if the following LMI holds for some $P_0 > 0$ and $P_1 > 0$:

$$\begin{bmatrix} A_0^{\mathrm{T}} P_0 + P_0 A_0 + P_1 & P_0 A_1 \\ A_1^{\mathrm{T}} P_0 & -P_1 \end{bmatrix} < 0, \tag{6}$$

which is known as two-dimensional (2-D) Lyapunov inequality (Agathoklis and Foda, 1989). This condition can be interpreted from two different points of view. *First*, according to the continuous-time bounded real lemma (Anderson and Vongpanitlerd, 1973), LMI (6) holds if and only if

$$\max_{\Re(s)=0} \sigma_{\max}(R_1 S(s) R_1^{-1}) < 1;\ R_1^{\mathrm{T}} R_1 = P_1. \tag{7}$$

In view of the relationships:

$$\max_{\Re(s)=0} \rho(S(s)) = \max_{\Re(s)=0} \rho\left(R_1 S(s) R_1^{-1}\right) \leq \max_{\Re(s)=0} \sigma_{\max}(R_1 S(s) R_1^{-1}), \tag{8}$$

it can be seen that (7), or equivalently (6), is more strict than condition (i) in Lemma 1. This frequency-domain interpretation can be found, e.g., in Boyd et al. (1994); Chen et al. (1995); Agathoklis and Foda (1989). *Second*, (6) can also be established from a time-domain point of view by using a *simple* LKF (Boyd et al., 1994). Both interpretations endow the condition in (6) with great power and extensibility in other more complicated problems of time-delay systems, especially, for system synthesis. This is an advantage of sufficient conditions similar to the 2-D Lyapunov inequality (6).

With respect to necessary and sufficient stability test, the inequality (8) becomes an equality if $R_1$ is allowed to depend on the frequency

$$\max_{\Re(s)=0} \rho(S(s)) = \max_{\Re(s)=0} \min_{\substack{R_1(s) \\ \text{invertible}}} \sigma_{\max}(R_1(s) S(s) R_1^{-1}(s)). \tag{9a}$$

As a consequence, $\max_{\Re(s)=0}\rho(S(s))<1$ is equivalent to

$$\exists P_1(s)>0,\text{ such that } S^*(s)P_1(s)S(s)-P_1(s)<0,\ \forall \Re(s)=0. \quad (9b)$$

Therefore the conservatism of condition (6) is caused by fixing $R_1(s)=R_1$ or $P_1(s)=P_1$ on the imaginary axis $\Re(s)=0$. In view of the discrete Lyapunov inequality $S^{\mathrm{T}}P_1S-P_1<0$, (9b) is called as *frequency-dependent 1-D Lyapunov inequality* (Agathoklis and Foda, 1989). Bliman (Bliman, 2002) has proposed an elegant LMI approach to construct a family of matrix functions $P_1(s)$ such that (9b) is satisfied.

In this paper, we will present a new method of constructing $P_1(s)$ to satisfy (9b), so as to obtain some new delay-independent stability conditions.

### 2.3. Stability conditions based on frequency-discretization

In this section, two stability conditions, based on conditions (i) and (ii) of Lemma 1, respectively, will be presented.

#### 2.3.1. The first stability condition

As commented above, the LMI condition (6) is conservative due to the fact that $R_1(s)$ in (9a) and $P_1(s)$ in (9b) are constrained to be constant matrices. In this paper, to reduce or eliminate this conservatism, $R_1(s)$ and $P_1(s)$ are chosen as *piecewise constant* functions of $\omega$. Because $A_0$ and $A_1$ are real matrices, we have $\sup_{\omega\in\mathbb{R}}\rho(S(\mathrm{j}\omega))=\max_{\omega\geq 0}\rho(S(\mathrm{j}\omega))$ , thus it is sufficient to only consider nonnegative frequencies. The set of nonnegative frequencies may be partitioned as follows,

$$\Omega^+\triangleq[0,\infty)=\textstyle\bigcup_{l=1}^{\kappa}\Omega_l, \quad (10)$$

where

$$\Omega_l=[\omega_{l-1},\omega_l],\ l=1,\ldots,\kappa-1,\ \Omega_\kappa=[\omega_{\kappa-1},\infty) \quad (11)$$

and $0=\omega_0<\omega_1<\cdots<\omega_{\kappa-1}<+\infty$. Scalars $\kappa$ and $\omega_l$, $l=1,2,\ldots,\kappa-1$ are to be determined later. Specifically, we constrain $P_1(s)$ and $R_1(s)$ in the following form:

$$P_1(\mathrm{j}\omega)=P_1^{(l)},\ R_1(\mathrm{j}\omega)=R_1^{(l)},\ \omega\in\Omega_l,\ l=1,2,\ldots,\kappa, \quad (12)$$

where $P_1^{(l)}$'s are Hermitian positive definite matrices, and $R_1^{(l)}$'s are nonsingular matrices that satisfy $R_1^{(l)\mathrm{T}}R_1^{(l)}=P_1^{(l)}$.

In each interval $\Omega_l$, one needs to test the existence of a constant nonsingular matrix $R_1^{(l)}$ to satisfy

$$\max_{\omega\in\Omega_l}\sigma_{\max}(R_1^{(l)}S(\mathrm{j}\omega)R_1^{(l)-1})<1, \quad (13)$$

or equivalently, a constant positive definite matrix $P_1^{(l)}$ to satisfy

$$S^*(\mathrm{j}\omega)P_1^{(l)}S(\mathrm{j}\omega)-P_1^{(l)}<0,\ \forall\omega\in\Omega_l. \quad (14)$$

First, we have the following theorem showing the existence of piecewise constant functions $P_1(s)$ and $R_1(s)$ if system (1) is strongly delay-independently stable.

**Theorem 1.** *System (1) is strongly delay-independently stable if and only if $\alpha(A_0)<0$ and there exist a positive integer $\kappa$, frequency intervals $\Omega_l$, $l=1,2,\ldots,\kappa$ satisfying (10) and nonsingular matrices $P_1^{(l)}\in\mathbf{H}_n$, $P_1^{(l)}>0$, $l=1,2,\ldots,\kappa$ to satisfy (14); or equivalently, $\alpha(A_0)<0$ and there exist $R_1^{(l)}\in\mathbb{C}^{n\times n}$ nonsingular such that (13) is satisfied for all $l=1,2,\ldots,\kappa$.*

**Proof.** The sufficiency is obvious from the above discussion.

To prove the necessity, suppose that system (1) is strongly delay-independently stable. Then according to Lemma 1, we have $\rho(S(\mathrm{j}\bar\omega))<1$, where $\bar\omega\in[0,\infty)$ is arbitrarily chosen and fixed. For this frequency $\bar\omega$,

$$\exists\bar R_1\text{ invertible, such that }\sigma_{\max}(\bar R_1S(\mathrm{j}\bar\omega)\bar R_1^{-1})<1.$$

Moreover, in view of $\alpha(A_0)<0$, we have

$$\lim_{\bar\omega\to\infty}\sigma_{\max}(\bar R_1S(\mathrm{j}\bar\omega)\bar R_1^{-1})=0.$$

In view of the continuity of $\sigma_{\max}(\bar R_1S(\mathrm{j}\bar\omega)\bar R_1^{-1})$ with respect to $\bar\omega$, the above equation implies that there exists a sufficiently large frequency $\bar\omega^*>0$ to satisfy

$$\sigma_{\max}(\bar R_1S(\mathrm{j}\bar\omega)\bar R_1^{-1})<1,\ \forall\bar\omega\geq\bar\omega^*. \quad (15)$$

Set $\omega_{\kappa-1}=\bar\omega^*$ and $R_1^{(\kappa)}=\bar R_1$, then we have

$$\max_{\omega\in\Omega_\kappa}\sigma_{\max}(R_1^{(\kappa)}S(\mathrm{j}\omega)R_1^{(\kappa)-1})<1. \quad (16)$$

It remains to be shown that (13) is satisfied for all $\omega\in[0,\bar\omega^*]$. To this end, since system (1) is strongly delay-independently stable, an invertible matrix function $\tilde R_1(\tilde\omega)$ exists such that

$$\sigma_{\max}(\tilde R_1(\tilde\omega)S(\mathrm{j}\tilde\omega)\tilde R_1^{-1}(\tilde\omega))<1,\ \forall\tilde\omega\in[0,\bar\omega^*] \quad (17)$$

or equivalently $\exists\tilde P_1(\tilde\omega)>0$ such that

$$S^*(\mathrm{j}\tilde\omega)\tilde P_1(\tilde\omega)S(\mathrm{j}\tilde\omega)-\tilde P_1(\tilde\omega)<0,\ \forall\tilde\omega\in[0,\bar\omega^*] \quad (18)$$

According to Bliman (2004), (18) must admit a polynomial solution of $\tilde P_1(\tilde\omega)$ in $\tilde\omega$, which implies that $\tilde R_1(\tilde\omega)$ and $\tilde R_1^{-1}(\tilde\omega)$ in (17), when chosen as $\tilde R_1(\tilde\omega)=\tilde P_1^{1/2}(\tilde\omega)$, both are *continuous* on $\tilde\omega\in[0,\bar\omega^*]$. Define a bivariate function

$$f(\tilde\omega,\omega)\triangleq\sigma_{\max}(\tilde R_1(\tilde\omega)S(\mathrm{j}\omega)\tilde R_1^{-1}(\tilde\omega))$$

on $[0,\bar\omega^*]^2$. Note that $f(\tilde\omega,\omega)$ is also continuous with respect to $\tilde\omega$ and $\omega$, and (17) implies that $f(\tilde\omega,\tilde\omega)<1$ for all $\tilde\omega\in[0,\bar\omega^*]$. Due to the continuity of $f(\tilde\omega,\omega)$ on a compact set $[0,\bar\omega^*]^2$ and according to Theorem 4.19 of Rudin (1976), $f(\tilde\omega,\omega)$ is uniformly continuous on $[0,\bar\omega^*]^2$. Therefore, there exists a sufficiently small constant $\varepsilon>0$ such that

$$f(\tilde\omega,\tilde\omega+\Delta\tilde\omega)<1,\ \forall(\tilde\omega,\Delta\tilde\omega)\in[0,\bar\omega^*]\times[-\varepsilon,\varepsilon]. \quad (19)$$

Now choose an integer $\kappa$ such that $m^*\triangleq\bar\omega^*/(\kappa-1)\leq 2\varepsilon$, and divide $[0,\bar\omega^*]$ equally into $\kappa-1$ sub-intervals. Correspondingly, $\Omega_l$, $l=1,\ldots,\kappa-1$ in (11) are $\Omega_l=[(l-1)m^*,lm^*]$. From (19), we have that, for all $l=1,\ldots,\kappa-1$,

$$1>\max_{\tilde\omega\in[0,\bar\omega^*]}\max_{\omega\in[\tilde\omega-\varepsilon,\tilde\omega+\varepsilon]}f(\tilde\omega,\omega)\geq\max_{\tilde\omega\in\Omega_l}\max_{\omega\in[\tilde\omega-\varepsilon,\tilde\omega+\varepsilon]}f(\tilde\omega,\omega).$$

Let $\tilde\omega=(l-1/2)m^*$ and choose $R_1^{(l)}=\tilde R_1((l-1/2)m^*)$, $l=1,\ldots,\kappa-1$, then we have

$$\Omega_l\subseteq[(l-1/2)m^*-\varepsilon,(l-1/2)m^*+\varepsilon].$$

Furthermore,

$$\begin{aligned}1&>\max_{\tilde\omega\in\Omega_l}\max_{\omega\in[\tilde\omega-\varepsilon,\tilde\omega+\varepsilon]}f(\tilde\omega,\omega)\\&\geq\max_{\omega\in[(l-1/2)m^*-\varepsilon,(l-1/2)m^*+\varepsilon]}\sigma_{\max}(R_1^{(l)}S(\mathrm{j}\omega)R_1^{(l)-1})\end{aligned}$$

$$\geq \max_{\omega\in\Omega_l}\sigma_{\max}(R_1^{(l)}S(\mathrm{j}\omega)R_1^{(l)-1}),\ l=1,\ldots,\kappa-1$$

which together with (16) implies the existence of $R_1^{(l)}$, $l = 1,2,\ldots,\kappa$ that satisfy (13). Let $P_1^{(l)} = R_1^{(l)\mathrm{T}}R_1^{(l)}$, $l = 1,2,\ldots,\kappa$, then (14) is also satisfied, and the proof is complete. ■

For each $l$, the conditions in (14) are the scaled bounded realness property of a continuous-time system $S(s)$ over a *finite* or *semi-infinite frequency* range $\Omega_l$. To make them numerically more tractable, we apply the GKYP lemma (Iwasaki and Hara, 2005) and propose the first necessary and sufficient LMI condition for the strong delay-independent stability of system (1).

**Theorem 2.** *System (1) is strongly delay-independently stable if and only if there exist a positive integer* $\kappa$*, frequency intervals* $\Omega_l$ *in (11),* $l=1,\ldots,\kappa$ *and nonsingular matrices* $P\in\mathbf{S}_n$, $P_0^{(l)}$, $P_1^{(l)}$, $Q_0^{(l)}\in\mathbf{H}_n$, $l=1,\ldots,\kappa$ *such that* $P>0$, $P_1^{(l)}>0$, $Q_0^{(l)}>0$, $l=1,\ldots,\kappa$ *and*

$$A_0^{\mathrm{T}}P+PA_0<0 \tag{20}$$

$$A^{\mathrm{T}}\left(\Phi_0\otimes P_0^{(l)}+\Psi_0^{(l)}\otimes Q_0^{(l)}\right)A+\Phi_1\otimes P_1^{(l)}<0,\ l=1,\ldots,\kappa \tag{21}$$

*where*

$$\begin{aligned}
&A\triangleq\begin{bmatrix}A_0 & A_1\\ \mathbf{I} & 0\end{bmatrix},\ \Phi_0\triangleq\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix},\ \Phi_1\triangleq\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\\
&\Psi_0^{(\kappa)}\triangleq\begin{bmatrix}1 & 0\\ 0 & -\omega_{\kappa-1}^2\end{bmatrix},\ \Psi_0^{(l)}\triangleq\begin{bmatrix}-1 & \mathrm{j}\omega_{\mathrm{c}}^{(l)}\\ -\mathrm{j}\omega_{\mathrm{c}}^{(l)} & -\omega_{l-1}\omega_l\end{bmatrix}\\
&\omega_{\mathrm{c}}^{(l)}\triangleq(\omega_{l-1}+\omega_l)/2,\ l=1,\ldots,\kappa-1.
\end{aligned}\tag{22}$$

**Proof.** According to the GKYP lemma (Theorem 4 of Iwasaki and Hara (2005)), (21) is equivalent to

$$\begin{bmatrix}S(\mathrm{j}\omega)\\ \mathbf{I}\end{bmatrix}^*\left(\Phi_1\otimes P_1^{(l)}\right)\begin{bmatrix}S(\mathrm{j}\omega)\\ \mathbf{I}\end{bmatrix}<0,\ \forall\omega\in\Omega_l,\ l=1,\ldots,\kappa$$

which is (14). Note that (20) is equivalent to (5). According to Theorem 1, it follows that the strong delay-independent stability of system (1) is equivalent to the existence of $P$, $P_0^{(l)}$, $P_1^{(l)}$, $Q_0^{(l)}$, $l=1,\ldots,\kappa$ such that $P>0$, $P_1^{(l)}>0$, $Q_0^{(l)}>0$, $l=1,\ldots,\kappa$ and (20), (21) are satisfied. The proof is thus complete. ■

*2.3.2. The second stability condition*

Theorem 2 is derived from condition (i) of Lemma 1. Starting from condition (ii) of Lemma 1, another stability condition can be obtained using the frequency-discretization technique.

From a frequency-domain point of view, condition (6) may be interpreted in the scaled positive realness sense of a discrete-time system (Agathoklis and Foda, 1989). In fact, from condition (ii) of Lemma 1 and the KYP lemma (Iwasaki and Hara, 2005), it follows that

$$\max_{z\in\partial\mathbb{D}}\alpha(Z(z))<0\ \Leftrightarrow\left\{\begin{array}{l}\exists R_0(z)\text{ invertible, such that}\\ \max_{z\in\partial\mathbb{D}}\lambda_{\max}(\mathrm{sym}(R_0(z)Z(z)R_0^{-1}(z)))<0\\ (\text{or }\exists P_0(z)>0,\text{ such that}\\ Z^*(z)P_0(z)+P_0(z)Z(z)<0,\ \forall z\in\partial\mathbb{D})\end{array}\right.\tag{23}$$

$$\Leftarrow\left\{\begin{array}{l}\exists R_0\text{ invertible, such that}\\ \max_{z\in\partial\mathbb{D}}\lambda_{\max}(\mathrm{sym}(R_0Z(z)R_0^{-1}))<0\\ (\text{or }\exists P_0>0,\text{ such that}\\ Z^*(z)P_0+P_0Z(z)<0,\ \forall z\in\partial\mathbb{D})\end{array}\right.$$

$$\Leftrightarrow\text{Condition (6).}$$

Hence, the conservatism of condition (6) is caused by constraining $R_0(z)$ and $P_0(z)$ to be constant on $z=\mathrm{e}^{\mathrm{j}\theta}$, $\theta\in[-\pi,\pi]$. Following the same spirit as the stability condition given in Theorem 1, we use piecewise constant functions $R_0(\mathrm{e}^{\mathrm{j}\theta})$ and $P_0(\mathrm{e}^{\mathrm{j}\theta})$ of $\theta$ to analyze condition (ii) in Lemma 1. Note that $Z(z)$ is a special discrete-time transfer function with $A_0$ and $A_1$ real matrices. Hence, it suffices to restrict the frequency to $[0,\pi]$. Partition $[0,\pi]$ to $\eta$ intervals

$$W\triangleq[0,\pi]=\bigcup_{l=1}^{\eta}W_l,\tag{24}$$

where $W_l=[\theta_{l-1},\theta_l]$, $l=1,2,\ldots,\eta$, and $0=\theta_0<\theta_1<\cdots<\theta_\eta=\pi$. Constrain $P_0(z)$ and $R_0(z)$ to be piecewise constant

$$P_0(\mathrm{e}^{\mathrm{j}\theta})=P_0^{(l)},\ R_0(\mathrm{e}^{\mathrm{j}\theta})=R_0^{(l)},\ \theta\in W_l$$

where $P_0^{(l)}$'s are positive definite matrices, and $R_0^{(l)}$'s are nonsingular matrices satisfying $R_0^{(l)\mathrm{T}}R_0^{(l)}=P_0^{(l)}$. The remaining work is to find $P_0^{(l)}$ or $R_0^{(l)}$ to test the following conditions for all $l=1,2,\ldots,\eta$,

$$\max_{\theta\in W_l}\lambda_{\max}(\mathrm{sym}(R_0^{(l)}Z(\mathrm{e}^{\mathrm{j}\theta})R_0^{(l)-1}))<0\tag{25}$$

or equivalently,

$$Z^*(\mathrm{e}^{\mathrm{j}\theta})P_0^{(l)}+P_0^{(l)}Z(\mathrm{e}^{\mathrm{j}\theta})<0,\ \forall\theta\in W_l.\tag{26}$$

The following two theorems show the existence of $R_0^{(l)}$ and $P_0^{(l)}$ satisfying (25) and (26) and how to find them by the LMI technique. Their proofs are similar to that of Theorems 1 and 2, respectively, and are thus omitted.

**Theorem 3.** *System (1) is strongly delay-independently stable if and only if there exist a positive integer* $\eta$*, frequency intervals* $W_l$, $l=1,2,\ldots,\eta$ *satisfying (24) and nonsingular matrices* $P_0^{(l)}\in\mathbf{H}_n$, $P_0^{(l)}>0$, $l=1,2,\ldots,\eta$ *to satisfy (26); or equivalently, there exist nonsingular matrices* $R_0^{(l)}\in\mathbb{C}^{n\times n}$ *such that (25) holds for all* $l=1,2,\ldots,\eta$.

**Theorem 4.** *System (1) is strongly delay-independently stable if and only if there exist a positive integer* $\eta$*, frequency intervals* $W_l$ *in (24),* $l=1,2,\ldots,\eta$ *and nonsingular matrices* $P_0^{(l)}$, $P_1^{(l)}$, $Q_1^{(l)}\in\mathbf{H}_n$, $l=1,2,\ldots,\eta$ *such that* $P_0^{(l)}>0$, $Q_1^{(l)}>0$, $l=1,2,\ldots,\eta$ *and*

$$A^{\mathrm{T}}\left(\Phi_0\otimes P_0^{(l)}\right)A+\Phi_1\otimes P_1^{(l)}+\Psi_1^{(l)}\otimes Q_1^{(l)}<0,\ l=1,2,\ldots,\eta\tag{27}$$

*where* $A$, $\Phi_0$ *and* $\Phi_1$ *are defined in (22), and*

$$\begin{aligned}
&\Psi_1^{(l)}\triangleq\begin{bmatrix}-2\cos\theta_{\mathrm{r}}^{(l)} & \mathrm{e}^{-\mathrm{j}\theta_{\mathrm{c}}^{(l)}}\\ \mathrm{e}^{\mathrm{j}\theta_{\mathrm{c}}^{(l)}} & 0\end{bmatrix}\\
&\theta_{\mathrm{c}}^{(l)}\triangleq(\theta_{l-1}+\theta_l)/2,\ \theta_{\mathrm{r}}^{(l)}\triangleq(\theta_l-\theta_{l-1})/2,\ l=1,2,\ldots,\eta.
\end{aligned}\tag{28}$$

**Remark 1.** The conditions in (20), (21) and (27) are LMIs and can be effectively tested via the existing numerical algorithms. If (21) is solvable for an integer $\kappa$, it is also solvable for $\kappa+1$ as long as the frequency sub-intervals $\Omega_l$'s are appropriately chosen as will be discussed in Subsection 2.4. Hence, there always is a way to reduce the conservatism of Theorem 2 by increasing $\kappa$. A similar observation can also be made to Theorem 4. Indeed, it has been shown that Theorems 2 and 4 can give exact stability testing, as long as the frequency intervals are sufficiently small.

*2.4. Test procedures based on frequency-discretization*

In Theorems 2 and 4, the values of $\kappa$ and $\eta$, and the specific discretization $\Omega_l$ and $W_l$ are unknown *a priori*. As commented in Remark 1, increasing $\kappa$, or making a frequency interval narrower, can reduce the conservatism. Hence, a natural idea is to gradually reduce the width of the frequency ranges for the LMIs until all the LMIs are satisfied and the frequency sets cover the entire frequency range. Following this idea, we provide a discretization strategy for finding $\kappa$ and $\Omega_l$'s, or $\eta$ and $W_l$'s such that system (1) is determined to be stable or unstable based on Theorems 2 and 4. For convenience, first define two auxiliary LMI conditions as follows:

$$P_1 > 0,\ Q_0 > 0 \text{ and } A^{\mathrm{T}}(\Phi_0 \otimes P_0 + \Psi_0 \otimes Q_0)A + \Phi_1 \otimes P_1 < 0 \quad (29)$$

$$P_0 > 0,\ Q_1 > 0 \text{ and } A^{\mathrm{T}}(\Phi_0 \otimes P_0)A + \Phi_1 \otimes P_1 + \Psi_1 \otimes Q_1 < 0 \quad (30)$$

where $A$, $\Phi_0$ and $\Phi_1$ are defined in (22), and

$$\Psi_0 = \begin{bmatrix} -1 & \mathrm{j}\omega_{\mathrm{c}} \\ -\mathrm{j}\omega_{\mathrm{c}} & -\underline{\omega}\bar{\omega} \end{bmatrix},\ \Psi_1 = \begin{bmatrix} -2\cos\theta_{\mathrm{r}} & \mathrm{e}^{-\mathrm{j}\theta_{\mathrm{c}}} \\ \mathrm{e}^{\mathrm{j}\theta_{\mathrm{c}}} & 0 \end{bmatrix}$$

and $\omega_{\mathrm{c}} = (\bar{\omega}+\underline{\omega})/2$, $\theta_{\mathrm{c}} = (\bar{\theta}+\underline{\theta})/2$, $\theta_{\mathrm{r}} = (\bar{\theta}-\underline{\theta})/2$ with $\bar{\omega}$, $\underline{\omega}$, $\bar{\theta}$ and $\underline{\theta}$ being known constant scalars. The LMIs in (29) and (30) are one of those in (21) and (27) over specific intervals $[\underline{\omega},\bar{\omega}]$ and $[\underline{\theta},\bar{\theta}]$, respectively. The test procedure is stated at follows.

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**Step 1** Choose an arbitrary nonsingular $\bar{R}_1$. Determine a sufficiently large $\bar{\omega}^*$, such that (15) is satisfied. Implement **Step 2** for the interval $[0,\bar{\omega}^*]$.

**Step 2** An interval is given when this step is implemented. Denote this interval as $[\underline{\omega},\bar{\omega}]$. Check the feasibility of the LMI (29), i.e., the existence of $P_0$, $P_1$ and $Q_0$ to satisfy (29) over $[\underline{\omega},\bar{\omega}]$.

- **a)** If feasible, then $\rho(S(\mathrm{j}\omega)) < 1$ for the given interval. Exit **Step 2** with the given interval. If not, continue with the following.
  - **b)** Set $\omega_{\mathrm{c}} = (\underline{\omega}+\bar{\omega})/2$. Check whether $\rho(S(\mathrm{j}\omega_{\mathrm{c}})) \geq 1$. If it is true, then declare the system not strongly delay-independently stable, and terminate the entire procedure.
  - **c)** If $\bar{\omega}-\underline{\omega} < \varepsilon$, then make a note that the condition cannot be determined in this interval, and exit **Step 2** with the given interval.
  - **d)** Implement **Step 2** for the interval $[\underline{\omega},\omega_{\mathrm{c}}]$.
  - **e)** Implement **Step 2** for the interval $[\omega_{\mathrm{c}},\bar{\omega}]$.

**Step 3** If in any one of the implementations of **Step 2-c)**, the condition cannot be determined, then declare the condition cannot be determined with the given accuracy level. Otherwise, declare the system is delay-independent stable.

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These test procedures are explained as follows.

- In **Step 2**, once we fail to directly check the condition $\rho(S(\mathrm{j}\omega)) < 1$ over the frequency range $[\underline{\omega},\bar{\omega}]$ using the LMI condition in (29), we divide the frequency range into two sub-intervals $[\underline{\omega},\omega_{\mathrm{c}}]$ and $[\omega_{\mathrm{c}},\bar{\omega}]$ to see if (29) is satisfied for each sub-interval. For practical programming, we conveniently implement this checking process in a recursive manner.
- The role of **Step 1** is to reduce the frequency range to be checked to a *finite* one. To find the frequency $\bar{\omega}^*$ such that $\rho(S(\mathrm{j}\omega)) < 1$ holds for all $\omega \in (\bar{\omega}^*,\infty)$, an upper bound of $\rho(S(\mathrm{j}\omega))$ can be estimated as
$$\rho(S(\mathrm{j}\omega)) \leq \sigma_{\max}((\mathrm{j}\omega I - A_0)^{-1}A_1) \leq (|\omega| - \sigma_{\max}(A_0))^{-1}\sigma_{\max}(A_1)$$
where it is assumed that $|\omega| > \sigma_{\max}(A_0)$. Hence, it suffices to choose
$$\bar{\omega}^* = \sigma_{\max}(A_0) + \sigma_{\max}(A_1) \quad (31)$$
to guarantee $\rho(S(\mathrm{j}\omega)) < 1$ for all $\omega \in (\bar{\omega}^*,\infty)$. Corresponding to (21) and **Step 1**, we may choose $P_1^{(\kappa)} = \bar{R}_1 = \mathbf{I}$.
- The checking process is stated specifically for Theorem 2. It is not difficult to adapt it to Theorem 4, for which $\omega$, $[0,\bar{\omega}^*]$ and (29) are replaced by $\theta$, $[0,\pi]$ and (30), respectively, and $\alpha\left(Z(\mathrm{e}^{\mathrm{j}\theta})\right) \geq 0$ is used instead of $\rho(S(\mathrm{j}\omega_{\mathrm{c}})) \geq 1$ in **Step 2-b)**.

**Remark 2.** When $\underline{\omega} = 0$, the explicit positive definiteness constraint $P_1 > 0$ can be removed in (29), because it has been implied by the last LMI in (29) (one can check the right lower $n \times n$ block of the last LMI in (29)). This happens when testing the first frequency interval through Theorem 2, and could be made use of to reduce the LMI size to be tested for Theorem 2.

**Remark 3.** We can also address systems that is weakly but not strongly delay-independently stable. For such systems, condition $\rho(S(\mathrm{j}\omega)) < 1$ holds only for $\omega > 0$ (Chen and Latchman, 1995). If $\rho(S(\mathrm{j}\omega)) = 1$ for $\omega = 0$, and $\frac{\mathrm{d}\rho(S(\mathrm{j}\omega))}{\mathrm{d}\omega} < 0$ for all $\omega \in [0,\omega_\varepsilon]$, where $\omega_\varepsilon$ is a sufficient small positive scalar, then the frequency-discritization method may be used to check the condition $\rho(S(\mathrm{j}\omega)) < 1$ over the frequency range $[\omega_\varepsilon,\infty)$.

**Remark 4.** The GKYP lemma can be used to deal with systems with kinds of finite frequency specifications, e.g., mitigate harmonics (Napoles et al., 2013). It is worth pointing out that by combining the control synthesis results based on the GKYP lemma (see Iwasaki and Hara (2007); Li and Gao (2014)) with the stability conditions presented above, new stabilization conditions for time-delay systems can also be derived. Note that there have been a lot of control synthesis results in the form of LMIs for time-delay systems since the seminal work Boyd et al. (1994). However, on one hand, most of the existing results focus on reducing conservatism in the *delay-dependent* aspect (see, e.g., Fridman and Shaked (2002); Wu et al. (2004); Palhares et al. (2005)), which, when dealing with the delay-independent aspect, actually can be reduced to the classic condition in (Boyd et al., 1994, Section 10.4). On the other hand, to the best of authors' survey and knowledge, many existing delay-independent LMI stabilization results (see, for example, de Oliveira and Geromel (2004); Ivaynescu et al. (2000); Niculescu (1998a)) are still based on a more conservative stability condition. Due to the utilization of the frequency-discretization idea, the delay-independent stabilization results based on the proposed stability conditions shall be less conservative than the one in (Boyd et al., 1994, Section 10.4). It should be noted that, different from the proposed stability conditions, any reasonably simple resulting stabilization ones are generally *sufficient but not necessary.*

## 3. A Numerical Example

This section provides a numerical example to illustrate the effectiveness of the proposed method. The LMI problems encountered in the proposed method will be solved by the free solver SDPT3 (Toh et al., 1999) via the parser YALMIP (Löfberg, 2004).

**Example 1.** Consider Example 3.3 of Gu et al. (2003) which is given by

$$\dot{x}(t) = A_0 x(t) + \beta A_1 x(t-d) \tag{32}$$

where $\beta$ is a real constant used for analysis, and

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -2 & -3 & -5 & -2 \end{bmatrix},\ A_1 = \begin{bmatrix} -0.05 & 0.005 & 0.25 & 0 \\ 0.005 & 0.005 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & -0.5 & 0 \end{bmatrix}. \tag{33}$$

It is known that the maximum of $\beta$ to maintain delay-independent stability is $\bar{\beta}^* = 1.21955$.

By applying the proposed method and some representative existing LMI based methods to test the stability of this example, we show in Table 1 the results on the achieved maximum of $\beta$ (denoted by $\bar{\beta}$), and the number of variables (NoV) and rows (NoR) of the LMI conditions for each method, where "SoS" denotes "sum-of-squares", "SA" means "state augmentation" and "FD" is "frequency discretization". It is seen that the proposed method in the paper (as well as those in Bliman (2002) and He et al. (2005)) can verify the delay-independent stability of this example for $\bar{\beta} = \bar{\beta}^*$, showing the improvement of the proposed method compared with the simple LMI condition in (6). Regarding the computational burden of those methods that confirm the exact margin $\bar{\beta} = \bar{\beta}^*$ for this example, Theorem 2 needs fewer variables, while Bliman (2002) is the most efficient one that has a condition of the smallest size and spends the least time among these methods.

**Remark 5.** In Zhang et al. (2003), a simple LMI condition was proposed for robust stability analysis of an uncertain matrix that in form is the same as condition (ii) of Lemma 1 but with $z$ in $Z(z)$ being a bounded *real*. Following the method in Zhang et al. (2003), a series of similar LMI conditions may also be established for robust analysis with a modulus-bounded *complex* parameter $z$ as in condition (ii) of Lemma 1. To this end, the scaling matrix $G_2$ in (37) in the reference should be set to zero. Interestingly, it is found that this modification along with $P_\Sigma > 0$ finally renders (37) in Zhang et al. (2003) coincide with the LMI condition in Bliman (2002). On the other hand, it is worth pointing out that Zhang et al. (2003) also provides an upper bound of the size of the condition, beyond which, the studied uncertain system is unstable if the condition is infeasible. This property is attractive because it is useful for instability test. Conditions with a similar property can also be found in Ebihara et al. (2006); Chesi (2013), which deal with Schur stability with a modulus-bounded uncertain complex parameter and (Schur or Hurwitz) stability with a bounded uncertain real parameter of the polynomial form, respectively. However, the stability conditions presented in this paper do not have this property, because there is no *a priori* knowledge on what the discretized frequency intervals should be, which is a drawback of the proposed method. For the considered problem, how to derive stability conditions of the same nature is an interesting topic, and the ideas presented in the mentioned references would be promising for this topic.

**Remark 6.** Although Theorem 2 and Theorem 4 both can exactly test delay-independent stability, there is no result that relates the $(P, Q)$ matrices in the conditions (21) to the others in (27). This is because the $(P, Q)$ matrices in the two sets of LMI conditions result from the discretization of *different* frequency variables. For this example with $\bar{\beta}^* = 1.21955$, we tried substituting the values of $P$ matrices obtained by Theorem 2 into the conditions of Theorem 4, but cannot find any solution to verify the stability of this example, although it is indeed stable. In addition, it should be pointed out that (20) is not explicitly included in Theorem 4 because it has been guaranteed by the satisfactoriness of (27) and (28).

**Remark 7.** As mentioned, the results in Table 1 show that Theorem 2 is more efficient than Theorem 4, but this depends on the specific example studied. For instance, if the same example for $\beta = 1.21955$ is investigated but with the third row of $A_1$ modified to $\begin{bmatrix} 0.7 & -0.2 & 0.5 & -0.18 \end{bmatrix}$, it is found that Theorem 2 with $\kappa = 4$ can ascertain the delay-independent stability, while Theorem 4 with $\eta = 2$ suffices to do this.

## 4. Conclusion

Strong delay-independent stability of linear time-invariant systems with state delay has been revisited in the paper, and a frequency-discretizing idea has been utilized to derive a series of new stability criteria. The proposed stability criteria are presented in terms of LMI, and can be easily tested by the existing numerical software. It is shown that the proposed stability criteria are necessary and sufficient for delay-independent stability with a single delay, as long as the discretization number goes to infinity. Numerical results have clearly demonstrated that the proposed method improves some classic simple LMI ones.

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Table 1: $\bar{\beta}$, NoV and NoR for different methods in Example 1

| Type | Method | $\bar{\beta}$ | NoV | NoR | Time (s) |
|---|---|---|---|---|---|
| Simple LMI | Condition (6)/Boyd et al. (1994) | 1.15057 | 20 | 12 | 0.36 |
| | Fridman (2002)/Fridman and Shaked (2002)/He et al. (2004) | 1.15057 | 78/52/68 | 20/16/20 | 0.50/0.40/0.49 |
| SoS + LMI | Papachristodoulou et al. (2009) ($L = 0$; $m = 2$) | 1.15057 | 20 | 12 | 0.68 |
| | Papachristodoulou et al. (2009) ($L = 0$; $m = 4$) | 1.15057 | 130 | 32 | 3.49 |
| SA + LMI | Bliman (2002) ($k = 1$) | 1.15057 | 20 | 12 | 0.36 |
| | Bliman (2002) ($k = 2$) | 1.21955 | 72 | 28 | 0.43 |
| | He et al. (2005) | 1.21955 | 136 | 32 | 0.54 |
| FD + LMI | Theorem 2 ($\kappa = 2$) | 1.21955 | 58 | 32 | 0.46 |
| | Theorem 4 ($\eta = 2$) | 1.19637 | $48 \times 2$ | $32 \times 2$ | 0.33 |
| | Theorem 4 ($\eta = 6$) | 1.21949 | $48 \times 6$ | $32 \times 6$ | 0.87 |
| | Theorem 4 ($\eta = 10$) | 1.21955 | $48 \times 10$ | $32 \times 10$ | 1.83 |


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